GUDLAVALLERU ENGINEERING COLLEGE
(An Autonomous Institute with Permanent Affiliation to JNTUK, Kakinada)
Seshadri Rao Knowledge Village, Gudlavalleru – 521 356.

Department of Electrical and Electronics Engineering

HANDOUT

On

INTEGRAL TRANSFORMS AND VECTOR CALCULUS

Vision

To be a pioneer in electrical and electronics engineering education and research, preparing students for higher levels of intellectual attainment, and making significant contributions to profession and society.

Mission

- To impart quality education in electrical and electronics engineering in dynamic learning environment and strive continuously for the interest of stake holders, industry and society.
- To create an environment conducive to student-centered learning and collaborative research.
- To provide students with knowledge, technical skills, and values to excel as engineers and leaders in their profession.

Program Educational Objectives

1. Graduates will have technical knowledge, skills and competence to identify, comprehend and solve problems of industry and society.
2. Graduates learn and adapt themselves to the constantly evolving technology to pursue higher studies and undertake research.
3. Graduates will engage in lifelong learning and work successfully in teams with professional, ethical and administrative acumen to handle critical situations.

Programme Specific Outcomes (PSO)

a. Apply the knowledge of circuit design, analog & digital electronics to the field of electrical and electronics systems

b. Analyze, design and develop control systems, industrial drives and power systems using modern tools

HANDOUT ON INTEGRAL TRANSFORMS AND VECTOR CALCULUS

Class & Sem. : I B.Tech – II Semester Year :2018-19
Branch : EEE Credits : 4

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1. **Brief History and Scope of the Subject**

   "MATHEMATICS IS THE MOTHER OF ALL SCIENCES", It is a necessary avenue to scientific knowledge , which opens new vistas of mental activity. A sound knowledge of engineering mathematics is essential for the Modern Engineering student to reach new heights in life. So students need appropriate concepts, which will drive them in attaining goals.

   Scope of mathematics in engineering study :

   Mathematics has become more and more important to engineering Science and it is easy to conjecture that this trend will also continue in the future. In fact solving the problems in modern Engineering and Experimental work has become complicated, time – consuming and expensive. Here mathematics offers aid in planning construction, in evaluating experimental data and in reducing the work and cost of finding solutions.

   The most important objective and purpose in Engineering Mathematics is that the students becomes familiar with Mathematical thinking and recognize the guiding principles and ideas "Behind the science" which are more important than formal manipulations. The student should soon convince himself of the necessity for applying mathematical procedures to engineering problems.

2. **Pre-Requisites**

   Basic Knowledge of Mathematics such as differentiation and Integration at Intermediate Level is necessary.

3. **Course Objectives:**
   - To gain the knowledge of Laplace and inverse transforms
   - To understand the concepts of fourier series and fourier transforms
   - To find the solutions of integral problems using vector concepts

4. **Course Outcomes:**

   Students will be able to

   CO1: apply Laplace transforms to find the solutions of ODE

   CO2: Express a function in Fourier series and in Fourier integral form.

   CO3: apply the concepts of vector differentiation and integration to the surface and volume integrals

5. **Program Outcomes:**

Program Outcomes:

Graduates of the Electrical and Electronics Engineering Program will

1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals and an engineering specialization for the solution of complex engineering problems.
2. **Problem analysis:** Identify, formulate, research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

3. **Design/development of solutions:** Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for public health and safety, and cultural, societal, and environmental considerations.

4. **Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

5. **Modern tool usage:** Create, select, and apply appropriate techniques, resources, and Modern engineering and IT tools, including prediction and modeling to complex engineering activities, with an understanding of the limitations.

6. **The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal, and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

7. **Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

8. **Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

9. **Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

10. **Communication:** Communicate effectively on complex engineering activities with the engineering community and with the society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

11. **Project management and finance:** Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

12. **Life-long learning:** Recognizes the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

Mapping of Course Outcomes with PO's & PSO's:

Subject Name	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**PSO1**	**PSO2**
CO: 1	3	2												
CO: 2	2	1												
CO: 3	2	1												

6. Prescribed Text Books

1. B.S.Grewal, Higher Engineering Mathematics : 42^{nd} edition, Khanna Publishers,2012 , New Delhi.
2. B.V Ramana, Higher Engineering Mathematics, Tata-Mc Graw Hill Company Ltd.

7. Reference Text Books

1. U.M.Swamy, A Text Book of Engineering Mathematics – I & II : 2nd Edition, Excel Books,2011, New Delhi.
2. Erwin Kreyszig, Advanced Engineering Mathematics : 8th edition, Maitrey Printech Pvt. Ltd, 2009, Noida.
3. Dr.T.K.V.Iyengar, Dr. B.Krishna Gandhi,S.Ranganatham and Dr.M.V.S.S.N.Prasad,Engineering Mathematics, Volume-I,II,III: 11th edition,S.chand Publishers,2012.New Delhi.

8. URLs and Other E-Learning Resources

Sonet CDs & IIT CDs on some of the topics are available in the digital library.

9. Digital Learning Materials:

- https://www.youtube.com/watch?v=2r_t8UaZosg&feature=youtu.be
- **https://www.youtube.com/watch?v=9UsQOxLKITc**
- **https://www.youtube.com/watch?v=x04dnqg-iPw**
- **https://www.youtube.com/watch?v=1JnayXHhjlg**
- **https://www.youtube.com/watch?v=AQhCGkK-hoA**
- **https://www.youtube.com/watch?v=o2kbrqQgzOE**

11. Lecture Schedule / Lesson Plan

S.No	TOPIC	No of. Periods	No of. Tutorials
	UNIT-I		
1	Laplace transforms of standard functions	1	1
2	Shifting Theorems	1	
3	change of scale	1	
4	Transforms of derivatives	1	
5	Transforms of integrals	1	1
6	Unit step function –Dirac's delta function	1	
7	Evaluation of Improper Integrals	2	
8	Review and conclusion	1	
	UNIT-II		
9	Inverse Laplace transforms	1	1
10	Inverse Laplace transforms by partial fractions	2	
11	Convolution theorem (with out proof).	1	1
12	Inverse Laplace transforms by Convolution theorem	2	
13	Solutions of ordinary differential equations using Laplace transforms	3	
	UNIT-III		
14	Fourier series	1	1
15	Determination of Fourier coefficients(with out proof)	2	
16	Fourier series in an arbitrary interval	3	

17	Half range sine and cosine series	2	1
18	Review and conclusion	1	
	UNIT-IV		
19	Fourier integral theorem & Problems	2	1
20	Properties of Fourier transform (without proofs)	3	
21	Fourier transform, sine and cosine transforms & Problems	1	
22	Inverse Fourier transforms	2	1
23	Review and conclusion	1	
	UNIT-V		
24	Vector differentiation	1	1
25	Gradient,divergence,curl	4	
26	Laplacian operator	2	1
27	Review and conclusion	2	
	UNIT-VI		
28	Line integral-work done	2	1
29	Surface integral-flux across the surface	2	
30	Volume integral	1	1
31	Green's Theorem	2	
32	Gauss Theorem	2	
33	Stoke's Theorem	2	
34	Review and conclusion	1	
TOTAL		56	12

12. **Seminar Topics**

- Modelling and solving higher order ODE for Electrical Circuits
- Modelling and solving PDE with Fourier Methods

INTEGRAL TRANSFORMS AND VECTOR CALCULUS

UNIT-I: LAPLACE TRANSFORMS

Objectives:

- To know the properties of Laplace transforms
- To know the Transform of one variable function to another variable function.
- To find the Laplace Transform of standard functions

Syllabus: Laplace transform of standard functions- Properties: Shifting Theorems, change of scale, derivatives, integrals, multiplication and division – Unit step function – Dirac Delta function ,Evaluation of improper integrals.

Course Outcomes:

The students is able to

- Calculate the Laplace transform of standard functions both from the definition and by using formulas
- Select and use the appropriate shift theorems in finding Laplace transforms.
- Evaluation of Improper integrals.

Introduction:

The Laplace Transformation

Pierre-Simon Laplace (1749-1827)

Laplace was a French **mathematician**, **astronomer**, and **physicist** who applied the Newtonian theory of gravitation to the solar system (an important problem of his day). He played a leading role in the development of the **metric system**.

The **Laplace Transform** is widely used in **engineering applications** (mechanical and electronic), especially where the driving force is discontinuous. It is also used in process control.

Laplace Transform (**LT**) is a powerful technique to solve differential equations whether ordinary or partial since it replaces the operations of calculus by operations of algebra.

***Definition*:** Let f be a function defined for t ≥ 0.We define laplace transform of f ,denoted by F(s) or L{f(t)) or $\overline{f}$ (s) as F(s) = L{f(t)}= $\int_0^\infty e^{-st} f(t)dt$ for those s for which the integral exists is called the Laplace Transform or one sided Laplace Transform.

Sufficient conditions for the existence of L.T:

1)f is piecewise continuous on the interval 0 ≤ t ≤ A for any A > 0.

2)f is of exponential order i.e., If $f(t)$ is defined for all $t > 0$ and there exists constants α and M such that $|f(t)| \le Me^{\alpha t}$ for all t.

- *Note (1):* One sided LTs are unilateral whereas two sided LTs are bilateral Laplace Transforms.

- *Note (2):* A two sided LT obtained by setting the other limit of integral as $-\infty$.

Laplace transforms of some elementary functions:

Let $f(t)=1$ then $L\{f(t)\}=L(1)=\frac{1}{s}, \quad s>0$

1. Let $f(t)=e^{at}$ then $L\{f(t)\}=L(e^{at})=\frac{1}{s-a}, \quad s>a$
2. Let $f(t)=e^{-at}$ then $L\{f(t)\}=L(e^{-at})=\frac{1}{s+a}, \quad s>-a$.
3. Let $f(t)=t^n$ then $L\{f(t)\}=L(t^n)=\frac{\Gamma(n+1)}{s^{n+1}}$.
4. Let $f(t)=\sin at$ then $L\{f(t)\}=L(\sin at)=\frac{a}{s^2+a^2}$, $s>0$.
5. Let $f(t)=\cos at$ then $L\{f(t)\}=L(\sin at)=\frac{s}{s^2+a^2}$, $s>0$.
6. Let $f(t)=\sinh at$ then $L\{f(t)\}=L(\sinh at)=\frac{a}{s^2-a^2}$, $s>|a|$.
7. Let $f(t)=\cosh at$ then $L\{f(t)\}=L(\sin at)=\frac{s}{s^2-a^2}$, $s>|a|$.

Properties of Laplace transform:

1. Laplace transform operator L is linear. Laplace transform of a linear combination (sum) of functions is the linear combination (sum) of Laplace transforms of the functions.
2. Change of scale property: When the argument t of f is multiplied by a constant k, s is replaced by s/k in $\bar{f}(s)$ or $F(s)$ and multiplied by $1/k$.
3. First shift theorem proves that multiplication of $f(t)$ by e^{at} amounts to replacement of s by $s-a$ in $\bar{f}(s)$.
4. Laplace transform of a derivative f' amounts to multiplication of $\bar{f}(s)$ by s (approximately but for the constant $-f(0)$).
5. Laplace transform of integral of f amounts to division of $\bar{f}(s)$ by s .
6. Laplace transform of multiplication of $f(t)$ by t^n amounts to differentiation of $\bar{f}(s)$ for n times w.r.t. s (with $(-1)^n$ as sign).

7. Division of $f(t)$ by t amounts to integration of $\bar{f}(s)$ between the limits s to ∞.
8. Second shift theorem proves that the L.T. of shifted function $f(t-a)u(t-a)$ is obtained by multiplying $\bar{f}(s)$ by e^{-as}.

Problems:

1) If f(t) = t^3+4t^2+5, then L[f(t)] = $\frac{\Gamma(4)}{s^4}+4\frac{\Gamma(3)}{s^3}+5\frac{\Gamma(2)}{s^2}=\frac{6}{s^4}+\frac{8}{s^3}+\frac{5}{s^2}$

2) Find Laplace transform of $\sin t \cos 2t$.

Solution: Let $f(t)=\sin t\cos 2t$

$$=\frac{1}{2}(\sin 3t-\sin t)$$

Apply LT on both sides, we have

$L(\sin t\cos 2t)=L\left[\frac{1}{2}(\sin 3t-\sin t)\right]=\frac{1}{2}L(\sin 3t)-\frac{1}{2}L(\sin t)$ (Using linearity property of LT)

$$=\frac{1}{2}\left(\frac{3}{s^2+9}\right)-\frac{1}{2}\left(\frac{1}{s^2+1}\right).$$

3) Find the LT of $e^{-4t}\sin 3t$.

Solution: Let $f(t)=\sin 3t$

By the definition of LT, $L\{\sin 3t\}=\frac{3}{s^2+a^2}$

Hence by first shifting theorem, $L\{e^{-4t}\sin 3t\}=\frac{3}{(s+4)^2+9}=\frac{3}{s^2+8s+25}$.

Laplace transforms of derivatives:

Statement: Let $f(t)$ be a real continuous function which is of exponential order and $f'(t)$ is sectionally continuous and is of exponential order. Then

$L\{f'(t)\}=s\bar{f}(s)-f(0)$ Where $\bar{f}(s)=L\{f(t)\}$.

In general,

$L\{f^{(n)}(t)\}=s^n\bar{f}(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-f^{(n-1)}(0).$

Laplace transforms of integrals:

Statement: Suppose $f(t)$ is a real function and $g(t)=\int_0^t f(u)du$ is a real function such that both $f(t),\ g(t)$ satisfy the conditions of existence of Laplace transform then

$$L\{g(t)\}=L\left[\int_0^t f(u)du\right]=\frac{\bar{f}(s)}{s} \quad \text{Where } \bar{f}(s)=L\{f(t)\}.$$

Laplace transform of the function $f(t)$ multiplied by t^n:

Statement: If $f(t)$ is sectionally continuous and is of exponential order and if

$$L\{f(t)\}=\bar{f}(s) \text{ then } L\{t^n f(t)\}=(-1)^n \frac{d^n \bar{f}(s)}{ds^n} \quad \text{where } n=1,2,\ldots.$$

Laplace transform of the function $f(t)$ divided by t^n:

If $L\{f(t)\}=\bar{f}(s)$ then $L\left(\frac{f(t)}{t}\right)=\int_0^\infty \bar{f}(s)ds$ provided $f(t)$ satisfy the condition of existence of LT and the right hand side integral exists.

4) Problem: Find the Laplace transform of $f(t)=t\cosh at$, using LT of derivatives.

Solution: We are given $f(t)=t\cosh at$.

It is known that $f^{'}(t)=a\cosh at+at\sinh at$ and

$$f^{''}(t)=2a\sinh at+a^2 t\cosh at$$

By applying LT on both sides, $L\{f^{''}(t)\}=2aL\{\sinh at\}+a^2L\{t\cosh at\}$

By the LT of derivatives, $s^2L\{f(t)\}-sf(0)-f^{'}(0)=2a\frac{a}{s^2-a^2}+a^2L\{t\cosh at\}$

Since $f(0)=0$ and $f^{'}(0)=1$, on simplification, we have

$$L\{t\cosh at\}=\frac{2a^2}{(s^2-a^2)^2}.$$

5) ***Problem:*** Find $L\left(\int_0^t ue^{-u}\sin 4u\,du\right)$.

Solution: Let $f(t)=\sin 4u$

By LT, $L\{\sin 4u\}=\frac{4}{s^2+4^2}=\frac{4}{s^2+16}$

By first shifting theorem, $L\{e^{-u}\sin 4u\} = \dfrac{4}{(s+1)^2+16} = \dfrac{4}{s^2+2s+17}$

Then by LT of $t^n f(t)$, $L\{ue^{-u}\sin 4u\} = -\dfrac{d}{ds}\left(\dfrac{4}{s^2+2s+17}\right) = \dfrac{4}{(s^2+2s+17)} = \bar{f}(s)$.

Therefore, the LT of integrals, we have

$$L\left(\int_0^t ue^{-u}\sin 4u du\right) = \frac{\bar{f}(s)}{s} = \frac{4}{s(s^2+2s+17)}.$$

6) ***Problem:*** Find $L\left(\dfrac{\sin at \cos bt}{t}\right)$.

Solution: Let $f(t) = \sin at \cos bt$

$$= \frac{1}{2}\left[\sin(a+b)t + \sin(a-b)t\right]$$

By applying LT on both sides,

$$L\{\sin at \cos bt\} = \frac{1}{2}\left[L\{\sin(a+b)t\} + L\{\sin(a-b)t\}\right]$$

$$= \frac{1}{2}\cdot\frac{(a+b)}{s^2+(a+b)^2} + \frac{1}{2}\cdot\frac{(a-b)}{s^2+(a-b)^2} = \bar{f}(s)$$

Now, by the LT of $\dfrac{f(t)}{t}$, $L\left\{\dfrac{\sin at \cos bt}{t}\right\} = \dfrac{1}{2}\displaystyle\int_s^\infty \dfrac{(a+b)}{k^2+(a+b)^2}ds + \dfrac{1}{2}\int_s^\infty \dfrac{(a-b)}{k^2+(a-b)^2}ds$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{k}{a+b}\right)\right]_s^\infty + \frac{1}{2}\left[\tan^{-1}\left(\frac{k}{a-b}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a+b}\right)\right] + \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a-b}\right)\right]$$

$$= \frac{1}{2}\cot^{-1}\left(\frac{s}{a+b}\right) + \frac{1}{2}\cot^{-1}\left(\frac{s}{a-b}\right).$$

Unit Step function:

Definition: Unit step function is defined as $U(t-a) = 0, \ t < a$

$$= 1, \ t > a$$

This function is also known as Heaviside unit function.

Laplace transform of Unit step function $U(t-a)$ is given by

$$L\{U(t-a)\} = \int_0^\infty e^{-st}U(t-a)dt = \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1dt = \int_a^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}.$$

Unit impulse function:

Definition: The unit impulse function denoted by $\delta(t-a)$ and is defined by

$$\delta(t-a)=\infty,\ t=a$$
$$=0,\ t\neq a$$

So that $\int_0^\infty \delta(t-a)dt=1 \quad (a\geq 0)$.

If a moving object collide with another object then for a short period of time large force is acting on the other body. To explain such mechanism we make use of unit impulse function, which is also called Dirac Delta function.

Evaluation of improper integrals by Laplace transforms:

7) ***Problem:*** Evaluate the integral $\int_0^\infty \frac{\cos at-\cos bt}{t}dt$.

Solution: Let $I=\int_0^\infty \frac{\cos at-\cos bt}{t}dt$.

$$=\int_0^\infty \frac{\cos at}{t}dt-\int_0^\infty \frac{\cos bt}{t}dt$$

Clearly the given integral is in the form $\int_0^\infty e^{-st}\frac{f(t)}{t}dt$ with $f_1(t)=\cos at$ and $f_1(t)=\cos bt$

We observe that $\int_0^\infty e^{-st}\frac{\cos at}{t}dt=\int_s^\infty L(\cos at)ds=\int_s^\infty \frac{s}{s^2+a^2}ds$ and

$$\int_0^\infty e^{-st}\frac{\cos bt}{t}dt=\int_s^\infty L(\cos bt)ds=\int_s^\infty \frac{s}{s^2+b^2}ds$$

$$\therefore \int_0^\infty e^{-st}\left(\frac{\cos at-\cos bt}{t}\right)dt=\int_0^\infty \frac{s}{s^2+a^2}ds-\int_0^\infty \frac{s}{s^2+b^2}ds=\int_0^\infty\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]ds$$

It is clear that the above integral reduces to I when $s=0$.

Therefore,

$$I=\int_0^\infty \frac{\cos at-\cos bt}{t}dt=\int_0^\infty\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]ds=\left[\frac{1}{2}\log(s^2+a^2)-\frac{1}{2}\log(s^2+b^2)\right]_0^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_0^\infty = \frac{1}{2}\left[\log 1 - \log\left(\frac{a^2}{b^2}\right)\right] = \frac{1}{2}\log\left(\frac{a^2}{b^2}\right).$$

Assignment/Tutorial Questions

SECTION-A

Objective Questions:

1. Find the laplace transform of $t^{5/2}$

   (a) $\frac{15}{8}\frac{\sqrt{\pi}}{s^{\frac{5}{2}}}$ (b) $\frac{15}{8}\frac{\sqrt{\pi}}{s^{\frac{7}{2}}}$ (c) $\frac{9}{4}\frac{\sqrt{\pi}}{s^{\frac{7}{2}}}$ (d) $\frac{15}{4}\frac{\sqrt{\pi}}{s^{\frac{7}{2}}}$

2. The Laplace transform of $f(t)=\sin^2 2t$ is ________________.
3. If $f(t)=e^{-3t}(\sin 2t+\cos 3t)$ then $L\{f(t)\}=$________________.
4. If $f(t)=\sin 2t\cos 3t$ then the Laplace transform of $f(t)$ is ________________.
5. If $f(t)=\dfrac{e^{2t}-e^{3t}}{t}$ then $L\{f(t)\}=$________________.
6. The value of $\int_0^\infty e^{-3t}t\,dt$ is ________________.
7. If $L\{f(t)\}=\bar{f}(s)=\dfrac{s}{s^2+1}$, $f(0)=0$ then $L\{f'(t)\}=$________________.
8. Assume $L\{f(t)\}=\sqrt{\dfrac{\Pi}{s-2}}$. Then $L\{e^{-4t} f(t)\}$ =

   a) $\sqrt{\dfrac{\Pi}{s-2}}$ b) $e^{-2s}\sqrt{\dfrac{\Pi}{s}}$ c) $e^{-2s}\sqrt{\dfrac{\Pi}{s+2}}$ d) $\sqrt{\dfrac{\Pi}{s+2}}$

9. If f(t) = sin(at) – atcos(at), then its Laplace transform is given by ________________
10. Find the laplace transform of $y(t)=e^{|t-1|}\,u(t)$.

    a) $\frac{2s}{1-s^2}e^{s}$

    b) $\frac{2s}{1+s^2}e^{-s}$

    c) $\frac{2s}{1+s^2}e^{s}$

    d) $\frac{2s}{1-s^2}e^{-s}$

11. Laplace transform of $t^2 \sin(2t)$

    a) $\left[\frac{12s^2-16}{(s^2+4)^4}\right]$

    b) $\left[\frac{3s^2-4}{(s^2+4)^3}\right]$

    c) $\left[\frac{12s^2-16}{(s^2+4)^6}\right]$

    d) $\left[\frac{12s^2-16}{(s^2+4)^3}\right]$

12. Find the laplace transform of e^t Sin(t).

a) $\frac{a}{a^2+(s+1)^2}$

b) $\frac{a}{a^2+(s-1)^2}$

c) $\frac{s+1}{a^2+(s+1)^2}$

d) $\frac{s+1}{a^2+(s+1)^2}$

13. Find the laplace transform of y(t)=e^t.t.Sin(t)Cos(t)

a) $\frac{4(s-1)}{[(s-1)^2+4]^2}$

b) $\frac{2(s+1)}{[(s+1)^2+4]^2}$

c) $\frac{4(s+1)}{[(s+1)^2+4]^2}$

d) $\frac{2(s-1)}{[(s-1)^2+4]^2}$

SECTION-B

Descriptive Questions:

1. Find the Laplace transform of $\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3$.
2. Find L[cos(at+b)]?
3. Find the Laplace transform of $f(t)=\begin{cases} e^t, 0<t<1 \\ 0 \quad\quad ,t>1 \end{cases}$
4. Find the Laplace transform of: (i). $f(t)=|t-2|+|t-3|, \quad \forall t \geq 0$
5. Evaluate $L[te^{-t}cos^2t]$.
6. Evaluate $L[\frac{cosat-cosbt}{t}]$.
7. Evaluate $L\left[\int_0^t \frac{e^t sint}{t} dt\right]$.
8. Evaluate L[tsint] and hence find $L[\int_0^t \int_0^t tsintdt\, dt]$.
9. Evaluate L[$\cos 3t.\cos 2t.\cos t$].
10. Derive the Laplace transform of Unit Step function and hence find$L[e^{t-3}u(t-3)]$].
11. Evaluate $\int_0^\infty e^{-t}\frac{sin^2t}{t}dt$
12. Evaluate $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt$, using Laplace transform.

SECTION-C

1. The Laplace Transform of cos(ωt) is $\frac{s}{s^2+\omega^2}$ then L(e^{-2t} cos4t) is

(GATE-2010)

(a) $\frac{s-2}{(s-2)^2+16}$ (b) $\frac{s+2}{(s-2)^2+16}$ (c) $\frac{s-2}{(s+2)^2+16}$ (d) $\frac{s+2}{(s+2)^2+16}$

2. I f F(s) is the L.T of f(t) then. L. T of $\int_0^t f(\tau)d\tau$ is **(GATE-2007)**

(a) $\frac{1}{s}F(s)$ (b) $\frac{1}{s}F(s)$ -f(0) (c)sF(s)-f(0) (d) $\int F(s)ds.$

3. L. T of functions tu(t) and u(t) sint are resp. **(GATE-1987)**

(a) $\frac{1}{s^2},\frac{s}{s^2+1}$ (b) $\frac{1}{s},\frac{1}{s^2+1}$ (c) $\frac{1}{s^2},\frac{1}{s^2+1}$ (d) $s,\frac{s}{s^2+1}$

4. The unilateral Laplace transform of f (t) is $\frac{1}{s^2+s+1}$ .The unilateral laplace transform of t f(t) is___________ **(GATE-2012)**

(a) $\frac{-s}{(s^2+s+1)^2}$ (b) $\frac{s}{(s^2+s+1)^2}$ (c) $\frac{-(2s+1)}{(s^2+s+1)^2}$ (d) $\frac{2s+1}{(s^2+s+1)^2}$

UNIT-II: INVERSE LAPLACE TRANSFORMS

Objectives:

- Understand the properties of Inverse Laplace transforms
- To solve Integral equations by using convolution theorem.
- To convert differential equations into algebraic equations using Laplace Transforms and inverse Laplace transforms.

Syllabus:

Inverse Laplace Transforms – by partial fractions - Convolution theorem(without proof).

Application: Solution of ordinary differential equations.

Subject Outcomes/Unit Outcomes:

After learning this unit, students will be able to:

- Find inverse Laplace Transforms of the functions $\bar{f}(s)$ to obtain $f(t)$.
- Apply convolution theorem to find the Laplace transform of product of functions.
- Use the method of Laplace transforms to solve systems of linear first-order differential equations.

Definition: Suppose $f(t)$ is a piecewise continuous function and is of exponential order. Let $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \bar{f}(s)$. The inverse Laplace Transform of $f(t)$ is defined as $L^{-1}\{\bar{f}(s)\} = f(t),$ where L^{-1}inverse operator of is L and vice-versa.

Inverse Laplace transforms of some elementary functions:

(1). $L^{-1}\left\{\frac{1}{s}\right\} = 1$ (2). $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ (3). $L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n$ (4). $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

(5). $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ (6). $L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$ (7). $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$, etc.

Properties of Inverse Laplace transform:

Linear property:

If $L^{-1}\{\bar{f}(s)\} = f(t), L^{-1}\{\bar{g}(s)\} = g(t)$, then $L^{-1}\{a\,\bar{f}(s) + b\,\bar{g}(s)\} = a\,f(t) + b\,g(t)$

Shifting Property:

If $L\{f(t)\} = \bar{f}(s)$ then $L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t),\ s > a$.

Change of scale property:

If $L\{f(t)\} = \bar{f}(s)$ then $L^{-1}\left\{\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)\right\} = f(at)$, then, $L^{-1}\{\bar{f}(as\} = \frac{1}{a}\bar{f}\left(\frac{t}{a}\right)$.

Problem: let $\bar{f}(s) = \frac{4s+4}{4s^2-9}$. Then by linearity property of inverse Laplace transforms (ILT),

$$L^{-1}\left\{\frac{4s+4}{4s^2-9}\right\} = L^{-1}\left\{\frac{4s}{4s^2-9}\right\} + L^{-1}\left\{\frac{4}{4s^2-9}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2-(3/2)^2}\right\} + L^{-1}\left\{\frac{1}{s^2-(3/2)^2}\right\} = \cosh\frac{3}{2}t + \frac{2}{3}\sinh\frac{3}{2}t$$

Problem: Find the I LT of $\frac{4}{(s+1)(s+2)}$.

Solution: Let $\bar{f}(s) = \frac{4}{(s+1)(s+2)}$

By applying partial fractions, we can rewrite $\bar{f}(s)$ as

$$\bar{f}(s) = \frac{4}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{As+2A+Bs+B}{(s+1)(s+2)}$$

Comparing like terms in the numerator, we obtain $A = 4$ and $B = -4$.

Therefore, $\bar{f}(s) = \frac{4}{(s+1)(s+2)} = \frac{4}{(s+1)} - \frac{4}{(s+2)}$

By applying linearity property, we have

$L^{-1}\{\bar{f}(s)\} = 4L^{-1}\left\{\frac{1}{s+1}\right\} - 4L^{-1}\left\{\frac{1}{s+2}\right\} = 4e^{-t} - 4e^{-2t}$. (This is the particular solution of an ordinary differential of order 2 whose roots of the auxiliary equation are -1 and -2.)

Problem: Find the ILT of $\frac{s+1}{s^2+s+1}$.

Solution: Consider $\bar{f}(s) = \frac{s+1}{s^2+s+1}$

$$= \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

By the linearity property of ILT, we have

$$L^{-1}\left(\frac{s+1}{s^2+s+1}\right)=L^{-1}\left(\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)+L^{-1}\left(\frac{1/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$=e^{-t/2}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t=e^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right].$$

Inverse Laplace Transforms of Derivatives:

Statement: If $L\{f(t)\}=\bar{f}(s)$ then $L^{-1}\left(\frac{d^n(\bar{f}(s))}{ds^n}\right)=(-1)^n t^n f(t).$

Inverse Laplace Transforms of Integrals:

Statement: If $L\{f(t)\}=\bar{f}(s)$ then $L^{-1}\left(\int_s^\infty \bar{f}(s)ds\right)=\frac{f(t)}{t}.$

Inverse Laplace Transform of type $s\bar{f}(s)$:

Statement: If $L\{f(t)\}=\bar{f}(s)$ and $f(0)=0$ then $L^{-1}\left(s\bar{f}(s)\right)=f^{'}(t)$

Inverse Laplace Transform of type $\frac{\bar{f}(s)}{s}$:

Statement: If $L\{f(t)\}=\bar{f}(s)$ then $L^{-1}\left(\frac{\bar{f}(s)}{s}\right)=\int_0^t f(t)dt$

Similarly, $L^{-1}\left(\frac{\bar{f}(s)}{s^2}\right)=\int_0^t\int_0^t f(t)dt$ and hence in general,

$L^{-1}\left(\frac{\bar{f}(s)}{s^n}\right)=\int_0^t\int_0^t\ldots\int_0^t f(t)dt\,dt\ldots dt$ (n-folded integral).

Problem: Evaluate $L^{-1}\left\{\frac{s}{\left(s^2+2^2\right)^2}\right\}$?

Solution: We know that $L^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$, then by derivative property of ILT,

we have $L^{-1}\left[\frac{-2s}{\left(s^2+a^2\right)^2}\right]=-\frac{t}{a}\sin at$, $\therefore L^{-1}\left\{\frac{s}{(s^2+2^2)^2}\right\}=\frac{t}{4}\sin 2t$.

Convolution Theorem: This is used to find inverse Laplace transforms of product of functions and the operation of convolution between two functions yields another function.

Definition: Suppose $L\{f(t)\}=\bar{f}(s)$ and $L\{g(t)\}=\bar{g}(s)$ then the convolution product of $f(t)$ and $g(t)$ is defined as:

$$f(t)*g(t)=\int_0^t f(r)g(t-r)dr$$, provided the integral exists.

Example: Using convolution theorem find the inverse Laplace transform of $\dfrac{s^2}{(s^2+4)(s^2+9)}$.

Solution: We are given $f(t)=\dfrac{s^2}{(s^2+4)(s^2+9)}$

The given function $f(t)$ can be rewritten as,

$$f(t)=\frac{s^2}{(s^2+4)(s^2+9)}=\frac{s}{(s^2+4)}\cdot\frac{s}{(s^2+9)}$$

By applying inverse Laplace transform, we have,

$$L^{-1}\{f(t)\}=L^{-1}\left\{\frac{s^2}{(s^2+4)}\cdot\frac{s^2}{(s^2+9)}\right\}$$

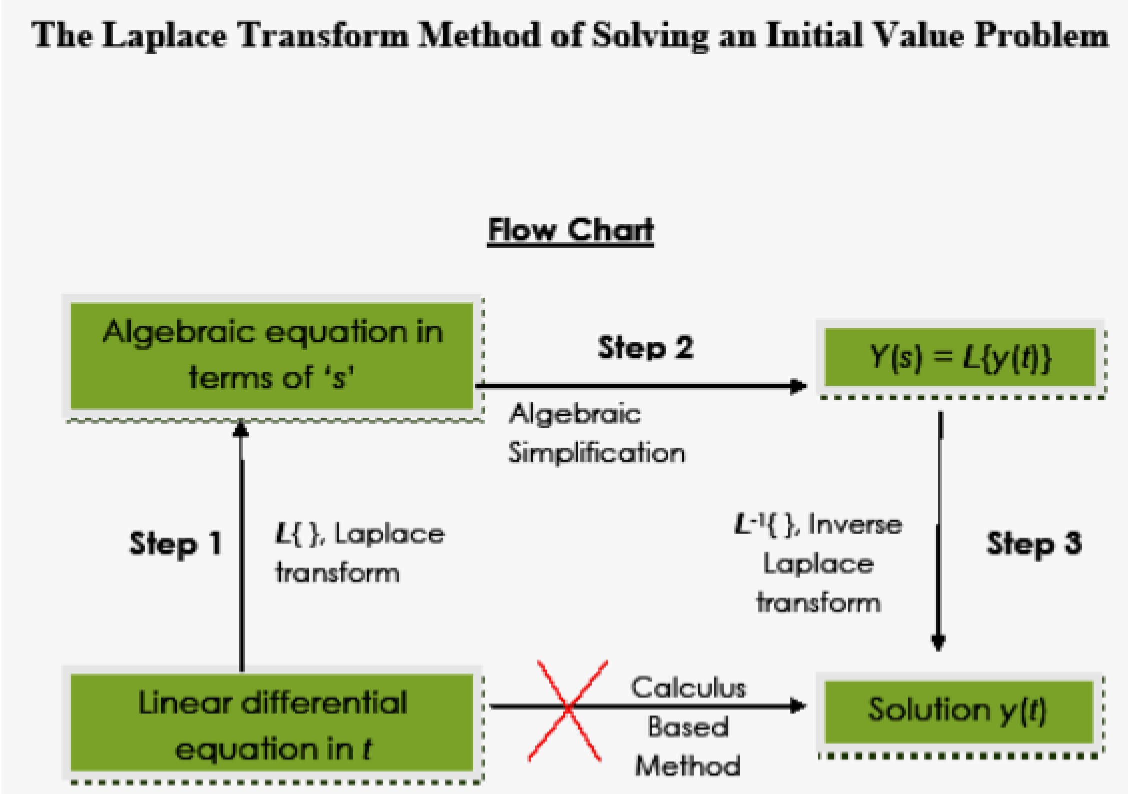
Hence by convolution theorem,

$L^{-1}\left\{\dfrac{s^2}{(s^2+4)}\cdot\dfrac{s^2}{(s^2+9)}\right\}=(\cos 2t)*(\cos 3t)$ since, $L^{-1}\left(\dfrac{s}{s^2+4}\right)=\cos 2t$ and $L^{-1}\left(\dfrac{s}{s^2+9}\right)=\cos 3t$

$$=\int_0^t[\cos 2u\cos 3(t-u)]du=\int_0^t\frac{1}{2}[\cos(3t-u)+\cos(5u-3t)]du$$

$$=\frac{1}{2}\left[\frac{\sin(3t-u)}{(-1)}\right]_0^t+\frac{1}{2}\left[\frac{\sin(5u-3t)}{5}\right]_0^t=\frac{-1}{2}[\sin 2t-\sin 3t]+\frac{1}{10}[\sin 2t+\sin 3t]$$

$$=\sin 2t\left(-\frac{1}{2}+\frac{1}{10}\right)+\sin 3t\left(\frac{1}{2}+\frac{1}{10}\right)=\frac{1}{5}(3\sin 3t-2\sin 2t).$$



Solution of Ordinary differential equation (An application):

Problem: Solve the differential equation $\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0;$ given $y(0)=y'(0)=0$ and $y''(0)=6$.

Solution: We are given the linear non-homogeneous differential equation with constant coefficients:

$$\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0 \text{ where } y=y(t) \text{ or } f(t)$$

Applying Laplace transform on both sides,

$$L\left(\frac{d^3y}{dt^3}\right)+2L\left(\frac{d^2y}{dt^2}\right)-L\left(\frac{dy}{dt}\right)-2L(y)=L(0)$$

$$\Rightarrow \left[s^3\bar{f}(s)-s^2f(0)-sy'(0)-y''(0)\right]+2\left[s^2\bar{f}(s)-sy(0)-y'(0)\right]-\left[s\bar{f}(s)-y(0)\right]-2\bar{f}(s)=0$$

$$\Rightarrow \bar{f}(s)\left[s^3+2s^2-s-2\right]-y(0)\left[s^2+2s-1\right]-y'(0)(s+2)-y''(0)=0$$

Substituting $y(0)=y'(0)=0$ and $y''(0)=6$, we get,

$$\bar{f}(s)(s^3+2s^2-s-2)-6=0$$

$$\Rightarrow \bar{f}(s)=\frac{6}{(s^3+2s^2-s-2)}$$

Now by applying inverse Laplace transform on both sides,

$$L^{-1}(\bar{f}(s))=L^{-1}\left(\frac{6}{s^3+2s^2-s-2}\right)=L^{-1}\left(\frac{6}{s^2(s+2)-(s+2)}\right)$$

$$f(t)=L^{-1}\left(\frac{6}{(s+2)(s+1)(s-1)}\right)$$

Consider $\bar{f}(s)=\dfrac{6}{(s-1)(s+1)(s+2)}=\dfrac{A}{(s-1)}+\dfrac{B}{(s+1)}+\dfrac{C}{(s+2)}$

On simplification we obtain $A=1, \quad B=-3, \quad C=2$

$$\therefore \quad L^{-1}(\bar{f}(s)) = f(t)=L^{-1}\left(\frac{1}{s-1}\right)-L^{-1}\left(\frac{3}{s+1}\right)+L^{-1}\left(\frac{2}{s+2}\right)$$

$$=e^{t}-3e^{-t}+2e^{-2t}$$

Hence, the solution of the given differential equation is $y(t)=e^{t}-3e^{-t}+2e^{-2t}$.

Problem: Solve the differential equation $t\dfrac{d^2y}{dt^2}+(1-2t)\dfrac{dy}{dt}-2y=0$ where $y(0)=1,\ y^{'}(0)=2$.

Solution: We are given the linear differential equation with variable coefficients:

$$t\frac{d^2y}{dt^2}+(1-2t)\frac{dy}{dt}-2y=0$$

Applying Laplace transform on both sides,

$$L\left(t\frac{d^2y}{dt^2}\right)+L\left((1-2t)\frac{dy}{dt}\right)-2L(y)=0$$

$$\Rightarrow -\frac{d}{ds}\left(s^2\bar{f}(s)-sf(0)-f^{'}(0)\right)+\left(s\bar{f}(s)-f(0)\right)+2\frac{d}{ds}\left(s\bar{f}(s)-f(0)\right)-2\bar{f}(s)=0$$

$$\Rightarrow \bar{f}^{'}(s)(2s-s^2)-s\bar{f}(s)=0$$

$$\Rightarrow \frac{\bar{f}^{'}(s)}{\bar{f}(s)}=-\frac{1}{s-2}$$

Integrating on both sides, we have,

$$\log \bar{f}(s)=-\log(s-2)+\log c$$

$$\Rightarrow \bar{f}(s)=\frac{c}{s-2}$$

By applying inverse Laplace transform on both sides,

$$L^{-1}(\bar{f}(s))=L^{-1}\left(\frac{c}{s-2}\right)$$

$$\Rightarrow f(t)=ce^{2t}$$

By using the initial condition, we have $c=1$.

Therefore, the particular solution of the differential equation is $f(t)=e^{2t}$.

IT-VC- UNIT-II
Assignment/Tutorial Questions
SECTION-A

(I) Objective Questions:

1. Time domain function of $s/(a^2 + s^2)$ is given by
   a) Cos(at)
   b) Sin(at)
   c) Cos(at)Sin(at)
   d) None of the above

2. The inverse laplace of $\frac{9}{s^5}$ is ________________.

3. The inverse laplace of $\frac{2}{(s+3)(s+1)}$ is _________________.

4. Find the inverse lapalce transform of $\frac{1}{(s^2+1)(s-1)(s+5)}$
   a) $\frac{1}{12} e^{t} - \frac{1}{13} Cos(-t) - \frac{1}{12} Sin(-t) - \frac{1}{156} e^{-5t}$
   b) $\frac{1}{12} e^{-t} - \frac{1}{13} Cos(t) - \frac{1}{12} Sin(t) - \frac{1}{156} e^{5t}$
   c) $\frac{1}{12} e^{t} - \frac{1}{13} Cos(t) - \frac{1}{12} Sin(t) - \frac{1}{156} e^{-5t}$
   d) $\frac{1}{12} e^{t} + \frac{1}{13} Cos(t) + \frac{1}{12} Sin(t) + \frac{1}{156} e^{-5t}$

5. Find $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} =$
   (a) sin2t + t/2 cos2t
   (b) ¼ sin2t+ tcos2t
   (c) ¼ sin2t+ t/2 cos2t
   (d) ¼ sin2t+1/4 tcos2t

6. The inverse laplace transform of $Y(S) = \frac{2se^{-s}}{1-s^2}$ is
   a) $-e^{-t+1} + e^{t-1}$
   b) $-e^{-t+1} - e^{t+1}$
   c) $-e^{-t+1} + e^{t+1}$
   d) $-e^{-t+1} - e^{t-1}$

7. The inverse lapace of $\frac{(s+1)}{((s+1)^2+4)((s+1)^2+1)}$ is
   a) $\frac{1}{3} e^{t}$ [Cos(t) – Cos(2t)]
   b) $\frac{1}{3} e^{-t}$ [Cos(t) + Cos(2t)]
   c) $\frac{1}{3} e^{t}$ [Cos(t) + Cos(2t)]
   d) $\frac{1}{3} e^{-t}$ [Cos(t) – Cos(2t)]

8. The inverse laplace of $\frac{s}{s^2+2s+3}$ is ___________________.

9. Given g(t)=t, h(t)= sint then g*h is________________

10. The inverse laplace of $\frac{3(s^2-1)^2}{2s^5}$ is ____________

11. $L^{-1}\left(\frac{s+2}{(s-2)^2}\right) =$
    (a) $e^{2t}(1+2t)$ (b) $te^{2t}(1+2t)$ (c) $(1+2t)$ (d) $t(1+2t)$

12. $L^{-1}\left(\frac{s+2}{s^2-2s+5}\right) =$

(a) $\cos 2t + \frac{3}{2}\sin 2t$ (b) $\sin 2t + \frac{3}{2}\cos 2t$ (c) $e^t \cos 2t + \frac{3}{2}e^t \sin 2t$ (d) $\cos 2t$

SECTION-B

(II) Descriptive Questions:

1. Suppose that y(t) satisfies the D.E $y^{11}-2y^{1}-y=1$ With $y(0)= -1, y^{1}(0)=1$ then find L(y(t)).
2. What is the convolution $e^t * \cos t$?
3. Consider the functions $f(t) = e^{t}$ and $g(t) = e^{-2t}$, $t \geq 0$. Compute $f * g$ in two different ways.
   a. By directly evaluating the integral.
   b. By computing L^{-1} [F(s)G(s)] where F(s) = L[f(t)] and G(s) = L[g(t)]
4. By convolution theorm find inverse laplace transform of

$$\text{I)}\left(\frac{6}{S(S^2+9)}\right) \qquad \text{II)}\ \frac{11S}{(S^2+121)^2}$$

5. Solve $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} - 12x = e^{3t}$, given that x(0) = 1 and $x^1(0)$ = -2 using Laplace transforms.
6. Find $L^{-1}\left(\log\left(\frac{s+1}{s-1}\right)\right)$
7. Find $L^{-1}\left(\frac{3S}{S^2+2S-8}\right)$
8. Obtain the inverse Laplace transform of $\frac{S}{(S^2+4)^2}$
9. Find $L^{-1}\left[\frac{S^2+S-2}{S(S-2)(S+3)}\right]$
10. Find $L^{-1}\left[\frac{3S+2}{(3S^2+4S+3)^2}\right]$
11. Find $L^{-1}\left[\frac{S+2}{S^2(S+1)(S-2)}\right]$

GATE PREVIOUS QUESTIONS

1. The function f(t) satisfies the differential equation $\frac{d^2 f}{dt^2} + f = 0$ and the auxiliary conditions, f(0)=0, $\frac{df}{dt}(0) = 4$. The Laplace transform of f(t) is given by **(GATE-2009)**

   (a) $\frac{2}{s+1}$ (b) $\frac{4}{s+1}$ (c) $\frac{4}{s^2+1}$ (d) $\frac{2}{s^2+1}$

2. The inverse Laplace transform of the function $F(s) = \frac{1}{s(s+1)}$ is given by **(GATE-2007)**

   (a) f(t)=sint (b) f(t) = e^{-t} sint (c) e^{-t} (d) 1- e^{-t}

3. The inverse Laplace transform of F(s) = s+1/(s^2+4) is **(GATE-2011)**

(a) cos2t+ sin2t (b) cos2t-(1/2) sin2t
(c) cos2t+(1/2)sin2t (d) cos2t-sin2t

4. Laplace transform of $f\left(t\right)\dfrac{1}{s^2\left(s+1\right)}$ then f(t) is **(GATE-2010)**

a) $t-1+e^t$ b) t + 1 +e^{-t} c) -1 +e^{-t} d) 2t +e^t

5. Inverse Laplace transform of $\dfrac{1}{s^2+s}$ is **(GATE-2009)**

a) 1 + e^t b) 1 – e^t c) 1 – e^{-t} d) 1 + e^{-t}

6. Inverse Laplace transform of $\dfrac{s+5}{\left(s+1\right)\left(s+3\right)}$ is **(GATE-1996)**

a) 2e^{-t} – e^{-3t} b) 2e^{-t} + e^{-3t} c) e^{-t} – 2e^{-3t} d) e^{-t} + 2e^{-3t}

7. Given $F(s)=\dfrac{S+2}{S^2+1}, G(S)=\dfrac{S^2+1}{(S+3)(S+2)}, h(t)=\int_0^t f(u)g(t-u)du, L(h(t)) is$

(GATE-2000)

a) $\dfrac{s^2+1}{s+3}$ b) $\dfrac{1}{s+3}$ c) $\dfrac{s^2+1}{\left(s+3\right)\left(s+2\right)}+\dfrac{s+2}{s^2+1}$ d) None

8. Given $f(t)=L^{-1}\left(\dfrac{3s+1}{s^3+4s^2+\left(k-3\right)s}\right) \lim_{t\to\infty} f(t)=1 \; then\, k$

a) 1 b) 2 c) 3 d) 4

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INTEGRAL TRANSFORMS AND VECTOR CALCULUS
Unit – 3
FOURIER SERIES

Objectives:

To introduce

- fourier series representation of a given function with period 2π (or) $2l$
- half range series representation of a given function with period π (or) l.

Syllabus:

Determination of Fourier coefficients (without proof) – Fourier series – even and odd functions – Fourier series in an arbitrary interval– Half-range sine and cosine series.

Outcomes:

Students will be able to

- expand the given function as Fourier series in the interval [c , c+2π]
- expand the given function as Fourier series in the interval [c , c+$2l$]
- expand the given function as Half-range Sine [or] Cosine series in the interval [0,l].
- write the expansions of $\frac{\pi^2}{8}, \frac{\pi^2}{6}, \frac{\pi^2}{12}, \ldots.$

Learning Material

Introduction:

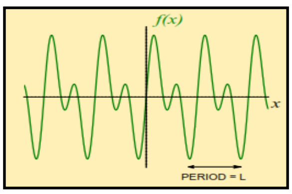
It became important to study the possibility of representation of the given function by infinite series other than power series. Since many phenomena like ***vibration of string, the voltages and currents in electrical networks, electro-magnetic signals, and movement of pendulum are periodic in nature***.

There is a possibility of representing a periodic function as an infinite series involving sinusoidal (sin x & cos x) functions. The French physicist J.B. Fourier announced in his work on heat conduction that an arbitrary periodic function could be expanded in a series of sinusoidal functions.

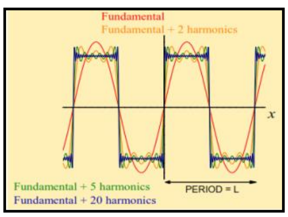
Thus, ***the aim of the theory of Fourier series is to determine the conditions under which the periodic functions can be represented as linear combinations of sine and cosine functions***.

Fourier methods give us a set of powerful tools for representing any periodic function as a sum of sines and cosines.

● A graph of **periodic** function $f(x)$ that has period L exhibits the same pattern every L units along the x-axis, so that $f(x+L) = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods)



One can even approximate a square-wave pattern with a suitable sum that involves a fundamental sine-wave plus a combination of harmonics of this fundamental frequency. This sum is called a **Fourier series**



Existence of Fourier series:

- **Dirichlet's Conditions :**

If a function f(x) is defined in $l \le x \le l + 2\pi$, it can be expanded as a Fourier series provided the following Dirichlet's conditions are satisfied

1. **f(x) is singe valued and finite in the interval (c , c + 2π)**
2. **f(x) is piece-wise continuous with finite number of discontinuities in (c ,c + 2π).**
3. **f(x) has finite number of maxima or minima in (c ,c + 2π).**

Note:

- These conditions are not necessary but only sufficient for the existence of Fourier series.
- If f(x) satisfies Dirichlet's conditions and f(x) is defined in (c, c + 2π), then f(x) need not be periodic for the existence of Fourier series of period 2π .
- If x = a is a point of discontinuity of f(x), then the value of the Fourier series at x = a is $\frac{1}{2}[f(a+)+f(a-)]$.

Basic Formulae to Solve Integration :

Continuous Derivatives

Continuous Integration

- Bracketing Method – [Through Examples]
  - $\int x^2.Sin\, nx\, dx = (x^2)\left(-\frac{Cos\, nx}{n}\right) - (2x)\left(-\frac{Sin\, nx}{n^2}\right) + (2)\left(\frac{Cos\, nx}{n^3}\right)$
  - $\int x.Cos\, nx\, dx = (x)\left(\frac{Sin\, nx}{n}\right) - (1)\left(-\frac{Cos\, nx}{n^2}\right)$
  - $\int x^2.Cos\frac{n\pi x}{L}dx = (x^2)\left(\frac{Sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (2x)\left(-\frac{Cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right) + (2)\left(\frac{Cos\frac{n\pi x}{L}}{\frac{n^3\pi^3}{L^3}}\right)$
- Spl. Formulae to Remember –
  - $\int e^{ax}.Sin\, bx\, dx = \frac{e^{ax}}{a^2+b^2}[a.Sin\, bx - b.Cosbx]$ $\int e^{ax}.Cosbx\, dx = \frac{e^{ax}}{a^2+b^2}[a.Cosbx + b.Sinbx]$
  - $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ [Here f(x) must be an **Even** function]
  - $\int_{-a}^{a} f(x)\,dx = 0$ [Here f(x) must be an **odd** function]
  - Values to Remember : Sin nπ = 0 & **Cos nπ =** $(-1)^n$

FULL RANGE FOURIER SERIES [Interval of length 2π]

The Fourier series for the function $f(x)$ in the interval **[c , c + 2π]** is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx.dx$ &

$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx.dx$

C = 0 ➔ **[0, 2 π]**

Remember these formulae as this carries 6M Problem.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Remember this formula as this carries 6M Problem.

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Cosnx\,dx$ & $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Sin\,nx\,dx$

C = $-\pi$ ➔ **[-π , π]**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).Cosnx\,dx$ & $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).Sin\,nx\,dx$

Examples:

1. Find the Fourier series to represent f(x) = x^2 in the interval (0,2π)

Sol. As the given interval is (0, 2π), Fourier series becomes -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Cosnx\,dx$ & $b_n =$

Step One : -

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}[(2\pi)^3 - 0] = \frac{8}{3}\pi^2$$

$$\Rightarrow a_0 = \frac{8}{3}\pi^2$$

Step Two : -

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\cos nx}_{v}\,dx$$

$$= \left[(x^2)\left(\frac{Sin\,nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{Sin\,nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{4}{n^2} \quad \left[\because \begin{matrix}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{matrix}\right] \qquad \Rightarrow a_n = \frac{4}{n^2}$$

Step Three : -

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\sin nx}_{v}\,d$$

$$= \left[(x^2)\left(-\frac{Cos\,nx}{n}\right) - (2x)\left(-\frac{Sin\,nx}{n^2}\right) + (2)\left(\frac{Cos\,nx}{n^3}\right)\right]_0^{2\pi}$$

Finally,

$$\therefore f(x) = x^2 = \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$\Rightarrow x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

2. Express the function $f(x) = \begin{cases} x & 0 < x < \pi \\ \pi & \pi < x < 2\pi \end{cases}$ as Fourier Series .

Sol. As the given interval is $(0, 2\pi)$, Fourier series becomes -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Cosnx\,dx$ & $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Sin\,nx\,dx$

STEP ONE

$$\begin{aligned} a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx &= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \frac{1}{\pi}\int_\pi^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_\pi^{2\pi} \pi\cdot dx \\ &= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{\pi}{\pi}\left[x\right]_\pi^{2\pi} \\ &= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) + \left(2\pi - \pi\right) \\ &= \frac{\pi}{2} + \pi \\ \text{i.e. } a_0 &= \frac{3\pi}{2}. \end{aligned}$$

STEP TWO

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_\pi^{2\pi} \pi\cdot\cos nx\,dx \\ &= \underbrace{\frac{1}{\pi}\left[\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{\sin nx}{n}\right]_\pi^{2\pi} \\ &= \frac{1}{\pi}\left[\frac{1}{n}\left(\pi\sin n\pi - 0\cdot\sin n0\right) - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right] \\ &\quad + \frac{1}{n}(\sin n2\pi - \sin n\pi) \\ \text{i.e. } a_n &= \frac{1}{\pi}\left[\frac{1}{n}\left(0-0\right) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] + \frac{1}{n}\left(0-0\right) \\ &= \frac{1}{n^2\pi}(\cos n\pi - 1), \quad \text{see TRIG} \\ &= \frac{1}{n^2\pi}((-1)^n - 1), \end{aligned}$$

$$\text{i.e. } a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_\pi^{2\pi} \pi\cdot\sin nx\,dx \\ &= \underbrace{\frac{1}{\pi}\left[\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)dx\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{-\cos nx}{n}\right]_\pi^{2\pi} \\ &= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right] - \frac{1}{n}(\cos 2n\pi - \cos n\pi) \\ &= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right] - \frac{1}{n}(1 - (-1)^n) \\ &= -\frac{1}{n}(-1)^n + \quad 0 \quad - \frac{1}{n}(1 - (-1)^n) \end{aligned}$$

Hence the Fourier series becomes ,

$$f(x) = \tfrac{1}{2}\left(\tfrac{3\pi}{2}\right) + \left(-\tfrac{2}{\pi}\right)\left[\cos x + 0\cdot\cos 2x + \tfrac{1}{3^2}\cos 3x + \ldots\right]$$
$$+\left(-1\right)\left[\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \ldots\right]$$

---

3. **Express** $f(x) = x - \pi$ **as Fourier series in the interval** $-\pi < x < \pi$

Sol Let the function $x - \pi$ be represented by the Fourier series

$$x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Then

Sep – 1

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx - \pi\int_{-\pi}^{\pi} dx\right]$$
$$= \frac{1}{\pi}\left[0 - \pi.2\int_{0}^{\pi} dx\right] \quad (\because x \text{ is odd function})$$
$$= \frac{1}{\pi}\left[-2\pi(x)_0^{\pi}\right]$$
$$= -2(\pi - 0) = -2\pi \text{ and}$$

Step - 2

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\cos nx.dx$$
$$= \frac{1}{x}\left[\int_{-\pi}^{\pi} x\cos nx.dx - \pi\int_{-\pi}^{\pi}\cos nx.dx\right]$$
$$= \frac{1}{\pi}\left[0 - 2\pi\int_{0}^{\pi}\cos nx.dx\right]$$

($\because x\cos nx$ is odd function and $\cos nx$ is even function)

$$\therefore a_n = -2\int_{0}^{\pi}\cos nx.dx$$
$$= -2\left(\frac{\sin nx}{n}\right)_0^{\pi}$$
$$= \frac{-2}{n}(\sin n\pi - \sin 0)$$
$$= \frac{-2}{n}(0-0) = 0 \text{ for } n = 1,2,3.......$$

Sep – 3

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx.dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx - \pi\int_{-\pi}^{\pi}\sin nx.dx\right]$$
$$= \frac{1}{\pi}\left[2\int_{0}^{\pi} x\sin nx.dx - \pi(0)\right]$$
$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) - (0+0)\right]$$
$$= \frac{-2}{\pi}\cos n\pi = \frac{-2}{n}(-1)^n$$
$$= \frac{2}{n}(-1)^{n+1}\ \forall n = 1,2,3.....$$

Substituting the values of a_0, a_n, b_n in (1),

We get ,

$$x-\pi = -\pi + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\pi} \sin nx$$

$$= -\pi + 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots \right]$$

---

4. Find the Fourier Series of the periodic function defined as

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ------ = \frac{\pi^2}{8}$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$ Then

Step 1 :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{o} (-\pi)\,dx + \int_{0}^{\pi} x\,dx \right]$$

$$= \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2} \right] = \frac{1}{\pi}\left[\frac{-\pi^2}{2} \right] = \frac{-\pi}{2}$$

Step 2 :

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{o} (-\pi)\cos nx.dx + \int_{0}^{\pi} x\cos nx.dx \right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right)_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{\pi n^2} \right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + \frac{1}{n^2} \right] = \frac{1}{\pi n^2}[(-1)^n - 1]$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi n^2}[(-1)^n - 1]}$$

Step 3 :

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

$$= \frac{1}{\pi}\left[\int_{\pi}^{0} (-\pi)\sin nx.dx + \int_{0}^{\pi} x\sin nx.dx \right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right)_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi \right]$$

$$= \frac{1}{n}(1 - 2\cos n\pi)$$

$$b_1 = 3, b_2 = \frac{-1}{2}, b_3 = 1, b_4 = \frac{-1}{4} \ldots\ldots\ldots\ldots$$

Substituting the values of a_0, a_n *and* b_n in (1), we get

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + --- \right) + \left(3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + -- \right)$$

Deduction: Put $x = 0$ in the above function $f(x)$, we get

Even and Odd Functions:-

A function $f(x)$ is said to be even $if\ f(-x)=f(x)$ and odd if $f(-x)=-f(x)$

Example :- $x^2, x^4+x^2+1, e^x+e^{-x}$ are even functions & $x^3, x, \sin x, \operatorname{cosec} x$ are odd functions.

Note 1 :-

1. Product of two even (or) two odd functions will be an even function
2. Product of an even function and an odd function will be an odd function

Note 2:- $\int_{-a}^{a} f(x)dx = 0$ when $f(x)$ is an odd function

$= 2\int_0^a f(x)dx$ When $f(x)$ is even function

Fourier series for even and odd functions

We know that a function $f(x)$ defined in $(-\pi,\pi)$ can be represented by the Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx$

And $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx.dx$

Case 1 :- when $f(x)$ is **even function**

Since $\cos nx$ is an even function,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

Case 2:- when $f(x)$ is an **odd function**

since $f(x)$ is an odd function

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=0$$

Examples:-

1. Expand the function $f(x) = x^2$ as a Fourier series in $(-\pi, \pi)$, hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ----= \frac{\pi^2}{12}$$

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$ ➔ ***f(x) is an even function.***

Hence in its Fourier series expansion, the sine terms are absent

$$\therefore x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$\therefore\ b_n = 0$

No need to find b_n.

Step 1 :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3}$$

Step 2 :

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx.dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx.dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 2\pi\frac{\cos nx}{n^2} + 2.0\right]$$

$$= \frac{4\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n$$

Substituting the values of a_0 *and* a_n , we get

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$
$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$$
$$= \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + --- \right)$$

Deductions:- Putting $x = 0$ in (4), we get

$$0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + --- \right)$$
$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---- = \frac{\pi^2}{12}$$

***FULL RANGE FOURIER SERIES** [Interval of length 2l]*

The Fourier series for the function $f(x)$ in the interval **[c, c + 2*l*]** is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)$$

Where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_c^{c+2l} f(x) Cos\frac{n\pi x}{l}\,dx$ $b_n = \frac{1}{l}\int_c^{c+2l} f(x)\,Sin\frac{n\pi x}{l}\,dx$

Remember this formula as this carries 6M Problem.

C = 0 ➔ **[0, 2*l*]**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)$$

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_0^{2l} f(x) Cos\frac{n\pi x}{l}\,dx$ & $b_n = \frac{1}{l}\int_0^{2l} f(x)\,Sin\frac{n\pi x}{l}\,dx$

Remember this formula as this carries 6M Problem.

C = $-l$ ➔ **[$-l$, l]**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)$$

Where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$, $a_n = \frac{1}{l}\int_{-l}^{l} f(x) Cos\frac{n\pi x}{l}\,dx$ & $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\,Sin\frac{n\pi x}{l}\,dx$

Examples:-

1. Express $f(x) = x^2$ as a Fourier series in $[-l, l]$

Sol $f(-x) = f(-x)^2 = x^2 = f(x)$

Therefore $f(x)$ is an even function

SEE FOR ***EVEN*** OR ***ODD*** FUNCTION AS THE INTERVAL IS FROM – VALUE TO + VALUE

Hence the Fourier series of $f(x)$ in $[-l,l]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$where\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$hence\ a_0=\frac{2}{l}\int_0^l x^2dx=\frac{2}{l}\left(\frac{x^3}{3}\right)_0^l=\frac{2l}{3}$$

$$also\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx\ =\frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[x^2\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]-2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l \qquad =\frac{2}{l}\left[2x\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l$$

Since the first and last terms vanish at both upper and lower limits

$$\therefore a_n=\frac{2}{l}\left[2l\frac{\cos n\pi}{n^2\pi^2/l^2}\right]=\frac{4l^2\cos n\pi}{n^2\pi^2}$$

$$=\frac{(-1)^n 4l^2}{n^2\pi^2}$$

Substituting these values in (1), we get

$$x^2=\frac{l^2}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n 4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

$$=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2}-\frac{\cos(2\pi x/l)}{2^2}+\frac{\cos(3\pi x/l)}{3^2}-----\right]$$

---

2. Find a Fourier series with period 3 to represent

$$f(x)=x+x^2\ \ in\ (0,3)$$

Sol. Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)\rightarrow(1)$

Here $2l=3$, $l=3/2$. Hence (1) becomes

$$f(x)=x+x^2$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3}+b_n\sin\frac{2n\pi x}{3}\right)\rightarrow(2)$$

$$Where\ a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$
$$= \frac{2}{3}\int_0^3 (x+x^2)\,dx = \frac{2}{3}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_0^3 = 9$$

$$and\ a_n = \frac{1}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$
$$= \frac{2}{3}\int_0^3 (x+x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

Using bracketing method, we obtain

$$a_n = \frac{2}{3}\left[\frac{3}{4n^2\pi^2 - 4n^2\pi^2}\right] = \frac{2}{3}\left(\frac{54}{9n^2\pi^2}\right) = \frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx \quad = \frac{2}{3}\int_0^3 (x+x^2)\sin\left(\frac{2n\pi x}{3}\right)dx \quad = \frac{-12}{n\pi}$$

Substituting the values of a's and b's in (2) we get

$$x + x^2 = \frac{9}{2} + \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

Half –Range Fourier Series (Interval of length *l*) ➔ [0,*l*]

Remember these formulae as this carries 6M Problem.

Part – B [3Q – (b)]

The Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.Cos\frac{n\pi x}{l}$$

Where

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x).Cos\frac{n\pi x}{l}$$

The sine series

$$f(x) = \sum_{n=1}^{\infty} b_n.Sin\frac{n\pi x}{l}$$

Where

$$b_n = \frac{2}{l}\int_0^l f(x).Sin\frac{n\pi x}{l}$$

Note:-

1) Suppose $f(x) = x\ in\ [0,\pi]$, it can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$
2) If $f(x) = x^2\ in\ [0,\pi]$, can have Fourier cosine series as well as sine series

Examples:-

1. Find the **half range sine** series for $f(x)=x(\pi-x)$ in $0<x<\pi$. Deduce that

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+----=\frac{\pi^3}{32}$$

Ans. The ***Fourier sine series*** expansion of $f(x)$ in (

Half range ➔

$(0,l)$ means $(0,\pi)$

$\therefore l=\pi$

$$f(x)=x(\pi-x)=\sum_{n=1}^{\infty}b_n\sin nx$$

$$where\ b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx.dx$$

$$hence\ b_n=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\sin nx.dx=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\sin nx.dx$$

$$=\frac{2}{\pi}\left[(\pi x-x^2)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{2}{n^3}(1-\cos n\pi)\right]$$

$$=\frac{4}{n\pi^3}\left(1-(-1)^n\right)$$

Hence

$$x(\pi-x)=\sum_{n=1,3,5...}\frac{8}{\pi n^3}\sin nx\ (or)$$

$$x(\pi-x)=\frac{8}{\pi}\left(\sin x+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+----\right)\rightarrow(1)$$

Deduction:- Putting $x=\frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2}\left(x-\frac{\pi}{2}\right)=\frac{8}{\pi}\left(\sin\frac{\pi}{2}+\frac{1}{3^3}\sin\frac{3\pi}{2}+\frac{1}{5^3}sin\frac{5\pi}{2}+--\right)$$

$$\frac{\pi^2}{4}=\frac{8}{\pi}\left[1+\frac{1}{3^3}\sin\left(\pi+\frac{\pi}{2}\right)+\frac{1}{5^3}\sin\left(2\pi+\frac{\pi}{2}\right)+\frac{1}{7^3}\sin\left(3\pi+\frac{\pi}{2}\right)+---\right]$$

Hence

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+----=\frac{\pi^3}{32}$$

---

2. Find the **half- range sine** series of $f(x)=1\ in\ [0,l]$

Ans. The Fourier sine series of $f(x)$ $in\ [0,l]$ is given by

$$f(x)=1=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$here\ b_n=\frac{2}{l}\int_0^{l}f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^{l}1.\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi / l} \right)_0^l$$

$$= \frac{2}{n\pi}\left[-\cos\frac{n\pi x}{l} \right]_0^l$$

$$= \frac{2}{n\pi}\left(-\cos n\pi + 1 \right)$$

$$= \frac{2}{n\pi}\left[(-1)^{n+1} + 1 \right]$$

$\therefore b_n = 0$ when n is even

$= \frac{4}{n\pi}$, when n is odd

Hence the required Fourier series is $f(x) = \sum_{n=1,3,5---}^{\infty} \frac{4}{n\pi} \sin\frac{n\pi x}{l}$.

3. Find the **half – range cosine** series expansion of $f(x) = \sin\left(\frac{\pi x}{l}\right)$ in the range $0 < x < l$

Sol. Half Range Cosine series in $(0,l)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.Cos\frac{n\pi x}{l}$

$$where\ a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}dx$$

$$= \frac{2}{l}\left[\frac{-\cos \pi x / l}{\pi / l} \right]_0^l$$

$$= \frac{2}{l}\left(\cos\pi - 1 \right) = \frac{4}{\pi}\ and$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^l \left[\frac{\sin(n+1)\pi x}{l} - \frac{\sin(n-1)\pi x}{l} \right]dx$$

$$= \frac{1}{l}\left[-\frac{-\frac{\cos(n+1)\pi x}{l}}{(n+1)\pi / l} + \frac{\cos(n-1)\pi x / l}{(n-1)\pi / l} \right]_0^l$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

When n is odd

$$a_n = \frac{1}{\pi}\left[\frac{-1}{n+1}+\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]=0$$

When n is even

$$a_n = \frac{1}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$= \frac{-4}{\pi(n+1)(n-1)}$$

$$\therefore \sin\left(\frac{\pi x}{l}\right)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos(2\pi x/l)}{1.3}+\frac{\cos(4\pi x/l)}{3.5}+----\right]$$

=======*========**=======*========**=======*========**=======*========

ITVC- UNIT-III

Assignment-Cum-Tutorial Questions

SECTION-A

Objective Questions

1. If $f(x)=\begin{cases}1+\frac{2x}{\pi} & -\pi\le x\le 0\\ 1-\frac{2x}{\pi} & 0\le x\le \pi\end{cases}$ Then $f(x)$ is ________ function [ ]

   a) Odd b) even c) periodic d) none

2. If the Fourier series for the function $f(x)$ defined in $[-\pi,\pi]$ then a_n = ________

3. The Fourier constant b_n *for* $f(x)=x\sin x$ *in* $[-\pi,\pi]$ is _____

4. If $f(x) = x^2$ in $(-l, l)$ then a_0 & b_1 are ______________

5. If $f(x) = |x|$ in $(-\pi, \pi)$ then a_1 & b_1 *are* _____________

6. In Fourier expansion of $f(x)=x+x^2$ in $(-\pi,\pi)$ the value of a_n is [ ]

   a) $\frac{2}{n^2}(-1)^4$ b) $\frac{4}{n^2}(-1)^n$ c) 0 d) none

7. If $f(x)=x\cos x$ *in* $(-\pi,\pi)$ then a_n is [ ]

   a) 1 b) 2 c) 3 d) 0

8. If $f(x)$ is expanded as a Fourier series in $(0,2\pi)$ then a_0 =__________ [ ]

   a) $\frac{1}{\pi}\int_0^{2\pi} f(x)dx$ b) $\frac{1}{\pi}\int_0^{\pi} f(x)dx$ c) $\frac{2}{\pi}\int_0^{2\pi} f(x)dx$ d) none

9. Fourier sine series for $f(x) = x$ in $(0, \pi)$ is _________________

10. If $f(x) = \sin x$ in $-\pi < x < \pi$ then a_0 =_______________

11. In Fourier series expansion of $f(x) = \cosh x$ in $(-4,4)$ the Fourier co efficient a_1 is _________________

12. If $f(x)$ is expanded as a Fourier series in $[0,2\pi]$ then b_n =________ [ ]

    a) $\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx.dx$ b) $\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx.dx$

c) $\frac{2}{\pi}\int_{0}^{2\pi} f(x)\sin nx.dx$ d) none

13.10. If $f(x)=1+\sin x$ in $(-1,1)$ is expressed as a Fourier series then the Value of b_n =______ [ ]

a) 0 b) 1 c) 2 d) none

SECTION-B

II) Level Two Questions:

1. Obtain Fourier Series for the function $f(x) = \begin{cases} x, & if\ 0 < x < \pi \\ 2\pi - x, & if\ \pi < x < 2\pi \end{cases}$

   And hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

2. Obtain the Fourier series to represent $x - x^2$ in $(-\pi, \pi)$ and deduce that

   $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$

3. If $f(x) = x^2$, $-l < x < l$. Obtian Fourier Series and deduce that

   $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$

4. Expand $f(x) = e^{-x}$ as a Fourier series in$(-1,1)$.

5. Obtain Fourier series to represent the function $f(x) = |x|$ in $(-\pi, \pi)$and

   deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

6. Obtain the Fourier series expansion of f(x) given that $f(x) = (\pi - x)^2\ in\ 0 < x < 2\pi$ and deduce that $1/1^2 + 1/2^2 + 1/3^2 + \ldots\ldots\ldots\ldots = \pi^2/6$

7. Find a Fourier series to represent the function $f(x) = e^x$ for $-\pi < x < \pi$ and hence derive a series for $\pi/sinh\pi$

8. Find the Fourier series of the periodic function $f(x) = \begin{cases} -\pi\ , -\pi < x < \pi \\ x\ ,\ 0 < x < \pi \end{cases}$

   Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \ldots\ldots\ldots. = \frac{\pi^2}{8}$

9. Find the half-range cosine series and sine series for $f(x) = x$ in $0 < x < \pi$ hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \ldots\ldots\ldots = \frac{\pi^2}{8}$

10. Find the Fourier series expansion for $f(x) = \begin{cases} 2, -2 < x < 0 \\ x, 0 < x < 2 \end{cases}$

11. Find the Fourier series expansion for the function $f(x) = x - x^2\ in\ (-1,1)$

12. Show that the Fourier series expansion of f(x) = 1 in 0<x<1 and f(x) = 2 in 1<x<3 with f(x+3) = f(x) is $\frac{5}{3} + \frac{9}{4\pi}\left[\frac{\sqrt{3}}{2}\cos\left(\frac{3\pi x}{2}\right) - \frac{\sqrt{3}}{4}\cos 3\pi x + \ldots\right] + \frac{9}{4\pi}\left[-\frac{3}{2}\sin\left(\frac{3\pi x}{2}\right) - \frac{3}{4}\sin 3\pi x + \ldots\right]$

    (DEC 2015)

13. Find the half-range cosine series for the function f(x)= $\begin{cases} kx, & 0 \le x \le \frac{l}{2} \\ k(l-x), & \frac{l}{2} \le x \le l \end{cases}$

14. Express f(x) = x as a half range sine series in 0 < x < 2.

15. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$

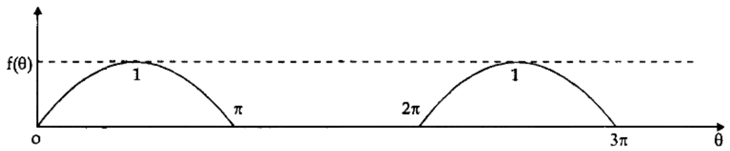
Hence show that $\sum_{n=1}^{\infty} \frac{1}{(2x-1)^2} = \frac{\pi^2}{8}$

SECTION-C

C. Questions testing the analyzing / evaluating ability of students

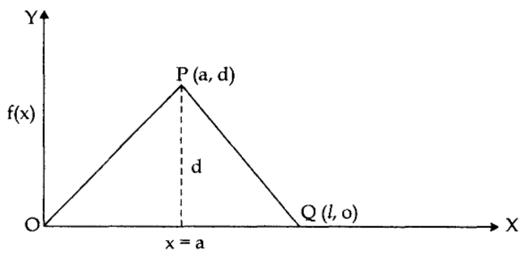
Level Three Questions:

1. An alternating current after passing through a rectifier has form $i = \begin{cases} I.\sin\theta & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$.



Find the Fourier series of the function.

2. Find the half period series for f(x) given in the range (0,L) by the graph OPQ as shown in the following fig.



Hint. $f(x) = \begin{cases} \frac{xd}{a}, & o < x < a \\ \frac{d(l-x)}{l-a}, & a < x < l \end{cases}$.

Gate Previous year Questions :

2016 Let $f(x)$ be a real, periodic function satisfying $f(-x) = -f(x)$. The general form of its Fourier series representation would be

(A) $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx)$

(B) $f(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$

(C) $f(x) = a_0 + \sum_{k=1}^{\infty} a_{2k} \cos(kx)$

(D) $f(x) = \sum_{k=0}^{\infty} a_{2k+1} \sin(2k+1)x$

2015

The signum function is given by

$$sgn(x) = \begin{cases} \frac{x}{|x|}; x \neq 0 \\ 0; x = 0 \end{cases}$$

The Fourier series expansion of $sgn(\cos(t))$ has

(A) only sine terms with all harmonics.
(B) only cosine terms with all harmonics.
(C) only sine terms with even numbered harmonics.
(D) only cosine terms with odd numbered harmonics.

Options :

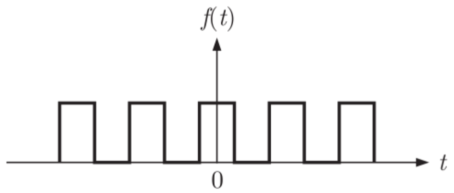
1. ✖ A
2. ✖ B
3. ✖ C
4. ✔ D

2012 Let $x(t)$ be a periodic signal with time period T, Let $y(t) = x(t - t_0) + x(t + t_0)$ for some t_0. The Fourier Series coefficients of $y(t)$ are denoted by b_k. If $b_k = 0$ for all odd k, then t_0 can be equal to

(A) $T/8$ (B) $T/4$

(C) $T/2$ (D) $2T$

2011 The Fourier series expansion $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t$ of the periodic signal shown below will contain the following nonzero terms



(A) a_0 and $b_n, n = 1,3,5,...\infty$ (B) a_0 and $a_n, n = 1,2,3,...\infty$

(C) a_0 a_n and $b_n, n = 1,2,3,...\infty$ (D) a_0 and a_n $n = 1,3,5,...\infty$

2010 The period of the signal $x(t) = 8\sin\left(0.8\pi t + \frac{\pi}{4}\right)$ is

(A) 0.4π s (B) 0.8π s

(C) 1.25 s (D) 2.5 s

2009

The Fourier Series coefficients of a periodic signal $x(t)$ expressed as $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$ are given by $a_{-2} = 2 - j1$, $a_{-1} = 0.5 + j0.2$, $a_0 = j2$, $a_1 = 0.5 - j0.2$, $a_2 = 2 + j1$ and $a_k = 0$ for $|k| > 2$

Which of the following is true ?

(A) $x(t)$ has finite energy because only finitely many coefficients are non-zero

(B) $x(t)$ has zero average value because it is periodic

(C) The imaginary part of $x(t)$ is constant

(D) The real part of $x(t)$ is even

2008

Let $x(t)$ be a periodic signal with time period T, Let $y(t) = x(t - t_0) + x(t + t_0)$ for some t_0. The Fourier Series coefficients of $y(t)$ are denoted by b_k. If $b_k = 0$ for all odd k, then t_0 can be equal to

(A) $T/8$ (B) $T/4$

(C) $T/2$ (D) $2T$

2007

A signal $x(t)$ is given by

$$x(t) = \begin{cases} 1, -T/4 < t \le 3T/4 \\ -1, 3T/4 < t \le 7T/4 \\ -x(t+T) \end{cases}$$

Which among the following gives the fundamental fourier term of $x(t)$?

(A) $\frac{4}{\pi}\cos\left(\frac{\pi t}{T} - \frac{\pi}{4}\right)$ (B) $\frac{\pi}{4}\cos\left(\frac{\pi t}{2T} + \frac{\pi}{4}\right)$

(C) $\frac{4}{\pi}\sin\left(\frac{\pi t}{T} - \frac{\pi}{4}\right)$ (D) $\frac{\pi}{4}\sin\left(\frac{\pi t}{2T} + \frac{\pi}{4}\right)$

****

INTEGRAL TRANSFORMS AND VECTOR CALCULUS
Unit – IV
FOURIER TRANSFORMS

Objectives:

To introduce

- Fourier transform of a given function and the corresponding inverse.
- Fourier sine and cosine transform of a given function and their corresponding inverses.
- Finite Fourier transforms of a given function and their corresponding inverses.

Syllabus:

Fourier integral theorem (only statement) – Fourier transform – sine and cosine transforms – properties – inverse Fourier transforms.

Outcomes:

Students will be able to

- Find the Fourier transform of the given function in infinite cases.
- Find the Fourier sine and cosine transforms of the given function in infinite cases.

Fourier Transforms are widely used to solve Partial Differential Equations and in various boundary value problems of Engineering such as Vibration of Strings, Conduction of heat, Oscillation of an elastic beam, Transmission lines etc.

Integral Transforms:

- The Integral transform of a function f(x) is defined as

$$I\{f(x)\} = \bar{f}(s) = \int_{x=x_1}^{x_2} f(x)K(s,x)dx$$

Where K(s,x) is a known function of s & x, called the 'Kernel' of the transform.

The function f(x) is called the Inverse transform of $\bar{f}(s)$

1.**Laplace Transform**: When K(s,x) = e^{-sx}

$$L\{f(x)\} = \bar{f}(s) = \int_0^\infty f(x)e^{-sx}dx$$

2.**Fourier Transform**: When K(s,x) = e^{isx}

$$F\{f(x)\} = \bar{f}(s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)e^{isx}dx$$

3.**Fourier Sine Transform**: When K(s,x)=Sinsx

$$F_s\{f(x)\} = \bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

4. **Fourier Cosine Transform**: When K(s,x)=CosSx

$$F_c\{f(x)\} = \bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

Fourier Integral Theorem:- If f(x) satisfies Dirichlet's conditions for expansion of Fourier series in (-c,c) and $\int_{-\infty}^{\infty}|f(x)|$ converges, then

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

is known as Fourier Integral of f(x)

Fourier Sine & Cosine Integrals:-

If f(x) satisfies Dirichlet's conditions for expansion of Fourier series in (-c,c) and $\int_{-\infty}^{\infty}|f(x)|$ converges,

- If f(t) is odd function then $f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda$

is called "Fourier sine Integral" .

- if f(t) is even function then $f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t\,dt\,d\lambda$

This is called "Fourier cosine Integral"

Complex form of Fourier Integral:-

- The complex form of Fourier integral is known as

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\lambda(t-x)}\,dt\,d\lambda$$

Problems:

1. using Fourier integral show that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}d\lambda, a,b>0$$

Solution: since the integrand on R.H.S contains sine term, we use Fourier sine integral formula.

We know that fouries sine integral for f(x) is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin px \int_0^\infty f(t)\sin pt\,dt\,dp$$

Replacing p with λ we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda$$

Here $f(x) = e^{-ax}-e^{-bx}$

$f(t) = e^{-at}-e^{-bt}$

substituting (2) in (1) , we get

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left(\int_0^\infty\left(e^{-at}-e^{-bt}\right)\sin\lambda t dt\right)d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\left\{\frac{e^{-at}}{\lambda^2+a^2}\left(-a\sin\lambda t-\lambda\cos\lambda t\right)\right\}_0^\infty-\left\{\frac{e^{-bt}}{\lambda^2+b^2}\left(-b\sin\lambda t-\lambda\cos\lambda t\right)\right\}_0^\infty\right]d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{\lambda^2+a^2}-\frac{\lambda}{\lambda^2+b^2}\right]d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda\left(b^2-a^2\right)}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}\right]d\lambda$$

$$e^{-ax}-e^{-bx}=\frac{2\left(b^2-a^2\right)}{\pi}\int_0^\infty\frac{\lambda\sin\lambda x}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}d\lambda$$

Fourier Transforms:-

- The fourier transform of a function f(x) is given by F(s) = $\int_{-\infty}^{\infty} f(x)e^{isx}dx$
- The inverse fourier transform of F(S) is given by f(x) = $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$

Fourier Sine transforms:-

- The Fourier sine transform of f(x) is defined as
  $F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sinsx\ dx$
- The inverse Fourier sine transform of $F_S(S)$ is defined as f(x) = $\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)sinsx\ ds$

here $F_s(s)$ is called **Fourier sine transform** of **f(x)** and **f(x)** is called **Inverse Fourier sine transform** of $F_s(s)$

Fourier Cosine transforms

- The Fourier cosine transform of f(x) is defined as
  $F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)cossx\ dx$
- The inverse Fourier cosine transform of $F_S(S)$ is defined as f(x) = $\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)cossx\ ds$

here $F_c(s)$ is called **Fourier cosine transform** of **f(x)** and **f(x)** is called **Inverse Fourier cosine transform** of $F_c(s)$

NOTE: 1. Some authors define F.T as follows

i) F(s) = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}dt$ ii) f(x) = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$

iii)F(s) = $\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx$ iv) f(x) = $\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} F(s)e^{isx}ds$

2.Some authors define Fourier sine & cosine transforms as follows

i) $F_s(s) = \int_0^\infty f(x) sinsx\, dx$ ii) f(x) = $\frac{2}{\pi}\int_0^\infty F_s(s) sinsx\, ds$

iii) $F_c(s) = \int_0^\infty f(x) cossx\, dx$ iv) f(x) = $\frac{2}{\pi}\int_0^\infty F_c(s) cossx\, ds$

Properties of Fourier Transforms:-

1. **Linearity Property**:- If $F_1(s)$ *and* $F_2(s)$ be the Fourier transforms of $f_1(x)$ *and* $f_2(x)$
2. respectively then f{a$f_1(x) + b\, f_2(x)\} = a\, F_1(s) + b\, F_2(s)$, *where a & b are constants*
3. .**Change of Scale Property**:- If F{f(x)} = F(s) then F{f(ax)} = $\frac{1}{a}F(\frac{s}{a})$
4. **Shifting Property**:- If F{f(x)} = F(s) then F{f(x-a)} = $e^{isa}F(s)$
5. **Modulation Property**:- If F{f(x)} = F(s) then F{f(x)cosax} = ½ {F(s+a)+F(s-a)}
6. If F{f(x)} = F(s) then F{f(-x)}= F(-s)
7. $\overline{F\{f(x)\}} = \overline{F(-s)}$
8. $\overline{F\{f(-x)\}} = \overline{F(s)}$
9. $F_c\{xf(x)\} = \frac{d}{ds}F_s\{f(x)\}$

Problem : Derive the relation between Fourier transform and Laplace transform.

Solution: consider $f(t) = \begin{cases} e^{-xt}g(t), t > 0,, \\ 0, t < 0 \end{cases}$

The fourier trasform of f(x) is given by

$$F(f(t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-xt}g(t)e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{(is-x)t}g(t)dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-pt}g(t)dt \quad \textbf{where p=x-is}$$

$$= \frac{1}{\sqrt{2\pi}}L(g(t)) \because \left[L\left(f(t) = \int_0^\infty e^{-st}f(t)dt\right)\right]$$

$\because$ **Fourier transform of** $f(t) = \frac{1}{\sqrt{2\pi}} \times$ **laplace transform of g(t)**

Problems:

Find the F.T of f(x) = $e^{-|x|}$

sol: Given f(x) = $e^{-|x|}$

$$= \begin{cases} e^{x}; x < 0 \\ e^{-x}; x > 0 \end{cases}$$

by definition, F{f(x)} = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\{\int_{-\infty}^{0} f(x)e^{isx}\,dx + \int_{0}^{\infty} f(x)e^{isx}\,dx\}$$

$$= \frac{1}{\sqrt{2\pi}}\{\int_{-\infty}^{0} e^{(1+is)x}\,dx + \int_{0}^{\infty} e^{(-1+is)x}\,dx\}$$

$$= \frac{1}{\sqrt{2\pi}}\ \{\left(\frac{e^{(1+is)x}}{1+is}\right)_{-\infty}^{0} + \left(\frac{-e^{-(1-is)x}}{1-is}\right)_{0}^{\infty}\ \}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+is} + \frac{1}{1-is}\right)$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{1}{1+s^2}$$

Problem: Find the Fourier transform of f(x) defined by $f(x) = \begin{cases} 1, \ if\ |x| < a \\ 0, \ if\ |x| > a \end{cases}$

And hence evaluate $\int_{0}^{\infty} \frac{\sin p}{p}\,dp$ and $\int_{-\infty}^{\infty} \frac{\sin ap\cos px}{p}\,dp$

Sol: We have F[f(x)]= $\int_{-\infty}^{\infty} e^{ipx} f(x)dx = \int_{-\infty}^{-a} e^{ipx} f(x)dx + \int_{-a}^{a} e^{ipx} f(x)dx + \int_{a}^{\infty} e^{ipx} f(x)dx$

$$= \int_{-a}^{a} e^{ipx}\,dx = \frac{2\sin ap}{p}$$

By the inversion formula, we know that f(x)= $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx} F(p)dp$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}\frac{2\sin ap}{p}\,dp = \begin{cases} 1, \ if\ |x| < a \\ 0, \ if\ |x| > a \end{cases}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \cos px\frac{2\sin ap}{p}\,dp - \frac{1}{2\pi}\int_{-\infty}^{\infty} \sin\ px\frac{2\sin ap}{p}\,dp = \begin{cases} 1, \ if\ |x| < a \\ 0, \ if\ |x| > a \end{cases}$$

Since the second integral is an odd function, $\int_{-\infty}^{\infty} \frac{\sin ap\cos px}{p}\,dp = \quad \pi,$

$$\begin{cases} 1, \ if\ |x| < a \\ 0, \ if\ |x| > a \end{cases}$$

Put x=0, we get, $\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin ap}{p}\,dp = \begin{cases} 1, \ if\ \ a > 0 \\ 0, \ if\ \ a < 0 \end{cases}$

$\int_{0}^{\infty} \frac{\sin\ p}{p}\,dp \ = \ \frac{\pi}{2},$ a>0

= 0, a<0

And put x=0 and a=1 then we get $\int_0^\infty \frac{\sin p}{p} dp = \frac{\pi}{2}$

Problem: Find the fourier sine transform of e^{-ax}, a>0 and hence deduce that $\int_0^\infty \frac{p \sin px}{a^2+p^2} dp$

Sol: $F_s\{f(x)\} = \int_0^\infty f(x) \sin px\, dx = \int_0^\infty e^{-ax} \sin px\, dx = \frac{p}{a^2+p^2}$

By the inversion formula, we know that f(x)= $\frac{2}{\pi}\int_0^\infty F_s\{f(x)\} \sin px\, dp$

$$= \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2} \sin px\, dp$$

$$\therefore \int_0^\infty \frac{p \sin px}{a^2+p^2} dp = \frac{\pi}{2} e^{-ax}$$

Problem : Find the Fourier sine Transform of $\frac{1}{x}$.

Sol: $F_s\{f(x)\} = \int_0^\infty f(x) \sin px\, dx = \int_0^\infty \frac{1}{x} \sin px\, dx$

Let px=θ

Pdx=dθ , θ:0 $\rightarrow \infty$

$$F_s\{f(x)\} = \int_0^\infty \frac{p}{\theta} \sin\theta \frac{d\theta}{p}$$

$$= \sqrt{\frac{\pi}{2}}$$

Problem : Find the Fourier cosine transform of f(x) = $\begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Solution : $F_c\{f(x)\} = \int_0^\infty f(x) \cos px\, dx = \int_0^a \cos x \cos px\, dx$

$$= \int_0^a \frac{\cos(p+1)x + \cos(p-1)x}{2} dx$$

$$= \left[\frac{\sin(p+1)a}{p+1} + \frac{\sin(p-1)a}{p-1}\right]\frac{1}{2}$$

ITVC - UNIT-IV

Assignment-Cum-Tutorial Questions

SECTION-A

Objective / Multiple choice Questions :

1. Fourier Cosine transform of f(x) is ____________________.

2. The inverse Fourier cosine transform of $f(x)$ is

   (a) $\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\, ds$ (b) $\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty F_c(s)\cos sx\, dx$

   (c) $\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos s\, dx$ (d) None.

3. Finite Fourier Sine transform of f(x) is _________________.

4. The inverse finite Fourier sine transform of $F_s(n)$ is

   (a) $\sum F_s(n)\sin\frac{n\pi x}{c}$ (b) $\frac{2}{c}\sum F_s(n)\sin\frac{n\pi x}{c}$

   (c) a and b (d) None.

5. Finite Fourier Cosine transform of f(x) is _______________.

6. $\int_0^\infty e^{-ax}\sin bx\, dx =$ __________________________

7. If $\tilde{f}(\alpha)$ is the Fourier transform of f(x), then the Fourier Transform

   of f(t-a) is

   a) $e^{ia\alpha}\tilde{f}(\alpha)$ b) $e^{ia\alpha}$ c) $e^{-ia\alpha}\tilde{f}(\alpha)$ d) $\frac{1}{a}\tilde{f}(\alpha/a)$

8. If $F\{f(x)\} = \tilde{f}(\alpha)$ *then* $F\{f(ax)\}$ *is*

   a) $\frac{1}{a}\tilde{f}(\alpha/a)$ b) $\tilde{f}(\alpha/a)$ c) $af(ax)$ d) None of these

9. If $F\{f(x)\} = \tilde{f}(\alpha)$ *then* $F\{f^{(n)}(x)\} =$ ....

   a) $\alpha^n\tilde{f}(\alpha)$ b) $(i\alpha)^n\tilde{f}(\alpha)$ c) $i^n\tilde{f}(\alpha/n)$ d) None of these

10. If $F\{e^{-|x|}\}=\sqrt{\frac{2}{\pi}}\frac{s}{1+s^2}$ then the value of the Fourier transform of $5e^{-4|x+2|}$ is

a) $\sqrt{\frac{2}{\pi}}.20e^{2i\alpha}\frac{1}{16+\alpha^2}$ b) $e^{2i\alpha}\frac{1}{16+\alpha^2}$ c) $\sqrt{\frac{2}{\pi}}\frac{20}{16+\alpha^2}$ d) None of these

SECTION-B

II) *Descriptive Questions*:

1. Find the Fourier transform of $f(x)=\begin{cases} e^{ikx} & a<x<b \\ 0 & x<a,\ x>b \end{cases}$.
2. Find the Fourier transform of $f(x)=\begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$, hence evaluate $\int_0^\infty \frac{S\,\text{int}}{t}dt$.
3. Find the Fourier transform of $f(x)=e^{-x^2/2}, -\infty<x<\infty$ [or] S.T Fourier transform of $e^{-x^2/2}$ is self reciprocal.
4. Find the Fourier transform of f(x) defined by $f(x)=\begin{cases} a^2-x^2, if\ |x|\le 1 \\ 0, \quad if\ |x|>1 \end{cases}$.

   And S.T. $\int_0^\infty \frac{\sin t - t\cos t}{t^3}dt=\frac{\pi}{4}$
5. Find the Fourier cosine and sine transform of $5e^{-2x}+2e^{-5x}$
6. Find the **a)** Fourier cosine and **b)** Fourier Sine transform of f(x) = e^{-ax} for x≥0 and a>0. And hence deduce the integrals known as "Laplace integrals" $\int_0^\infty \frac{\cos\alpha x}{\alpha^2+a^2}d\alpha$ and $\int_0^\infty \frac{\alpha.Sin\alpha x}{\alpha^2+a^2}d\alpha$
7. Find the inverse Fourier cosine transform f(x) if

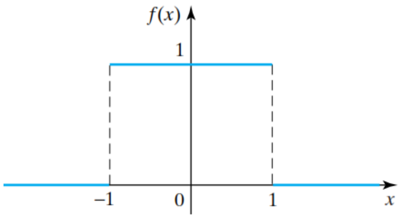
   $$F_c(\alpha)=\begin{cases} \frac{1}{2a}(a-\frac{\alpha}{2}), & \alpha<2a \\ 0, & \alpha\ge 2a \end{cases}$$
8. Find Fourier sine transform f(x)=$e^{-|x|}$ & hence find $\int_0^\infty \frac{x.\sin mx}{1+x^2}dx$
9. Find the Fourier cosine and sine transform of xe^{-ax}.
10. Find the Fourier sine and cosine transforms of $f(x)=\frac{e^{-ax}}{x}$. S.T.

    $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}Sinsxdx=\tan^{-1}\left(\frac{s}{a}\right)-\tan^{-1}\left(\frac{s}{b}\right).$$

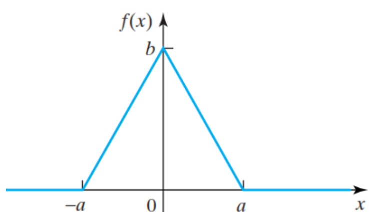
SECTION-C

C. Questions testing the analyzing / evaluating ability of students

Find the Fourier integral representation of the following functions.

1. 

The rectangular



pulse function

Hint :

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

2. The triangular function

Hint

$$f(x) = \begin{cases} 0, & |x| > a \\ b\left(1 + \frac{x}{a}\right), & -a \le x \le 0 \\ b\left(1 - \frac{x}{a}\right), & 0 \le x \le a \end{cases}$$

Gate Previous years Questions:

The value of the integral

$$\int_{-\infty}^{\infty} \sin c^2(dt) \quad is \_\_\_\_\_\_\_.$$

[**2014]**

ANS. 0.2

. Let $g(t) = e^{-\pi t^2}$, and $h(t)$ is filter marched to $g(t)$. If $g(t)$ is applied as input to $h(t)$, then the Fourier transform of the output is

(a) $e^{-\pi t^2}$

(b) $e^{-\pi f^2/2}$

(c) $e^{-\pi|f|}$

(d) $e^{-2\pi f^2}$

[2013]

ANS. (D)

.The Fourier transform of a signal h(t) is $H(j\omega) = (2\cos\omega)(\sin 2\omega)/\omega$. The value of h(0) is

(a) 1/4

(b) 1/2

(c) 1

(d) 2

[2012]

ANS. (C)

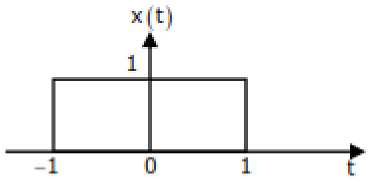
x(t) is a positive rectangular pulse from *t* = -1 to *t* = +1 with unit height as shown in the figure. The value of $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$ {where $X(\omega)$ is the Fourier transform of x(t)} is

(A) 2

(B) 2π

(C) 4

(D) 4π



[2010]

INTEGRAL TRANSFORMS AND VECTOR CALCULUS

Learning Material

UNIT V: Vector Differentiation

INTRODUCTION

Course Objectives:

• To introduce concept of gradient
• To introduce concept of divergence and
• To introduce concept of curl

Course Outcomes:

• To understand the physical interpretations of gradient
• To apply the gradient in various physical and engineering problems.
• To understand the physical interpretations of divergence
• To understand the physical interpretations of curl

Learning Material

❖ **UNIT V: Vector Differentiation and Vector Operators**

❖ **Scalar and vector point functions:** Consider a region in three dimensional space. To each point $p(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a ***scalar point function***. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$we associate a unique vector $\bar{f}\,(x,y,z)$. $\bar{f}$ is called a ***vector point function.***

Examples:

1. For example take a heated solid. At each point p(x,y,z)of the solid, there will be temperature $T(x,y,z)$. This T is a scalar point function.
2. Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position $p(x,y,z)$ in space, it will be having some speed, say, v. This **speed** v is a scalar point function.
3. Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.
4. In a magnetic field, at any point P(x,y,z) there will be a magnetic force $\bar{f}\,(x,y,z)$. This is called magnetic force field. This is also an example of a vector point

function. The students will come across several scalar and vector point functions in their respective subjects of study.

❖ **Tangent vector to a curve in space.**

Consider an interval [a,b].

Let x = x(t),y=y(t),z=z(t)be continuous and derivable for a≤ t ≤b.

Then the set of all points (x(t),y(t),z(t)) is called a curve in a space.

Let A = (x(a),y(a),z(a)) and B = (x(b),y(b),z(b)).

Then A, B are called the end points of the curve and if A =B, the curve in said to be a closed curve.

If $\bar{r} = (x(t), y(t), z(t))$ at the point P on the curve $(i.e., \bar{r} = \overline{OP})$ then $\frac{d\bar{r}}{dt}$ will be a tangent vector to the curve at P. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector and Suppose arc length AP = s. if we take the parameter as the arc length , we can observe that $\frac{d\bar{r}}{ds}$ is unit tangent vector at P to the curve.)

❖ **<u>VECTOR DIFFERENTIAL OPERATOR</u>**

Def. The vector differential operator ∇(read as del) is defined as $\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$.

This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator.

We will define now some quantities known as "**gradient", "divergence"** and **"curl"** involving the operator ∇.

We must note that this operator has no meaning by itself unless it operates on some function suitably.

<u>GRADIENT OF A SCALAR POINT FUNCTION</u>

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ and is denoted by gradϕ or ∇ϕ.

i.e., $\nabla\phi = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Properties:

(1) If f and g are two scalar functions then grad(f ±g)= grad f ± grad g

(2) The necessary and sufficient condition for a scalar point function 'f' to be constant is that $\nabla f = \bar{O}$

(3) grad(fg) = *f* (grad g)+g(grad f)

(4) If c is a constant, grad (c*f*) = *c*(grad *f*)

(5) grad $\left(\frac{f}{g}\right) = \frac{g(grad\ f) - f(grad\ g)}{g^2}$, $(g \neq 0)$

(6) Let r = $x\bar{i}+y\bar{j}+z\bar{k}$. Then $d\bar{r} = (dx)\bar{i}+(dy)\bar{j}+(dz)\bar{k}$. if ϕ is any scalar point function, then $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(\bar{i}dx + \bar{j}dy + \bar{k}dz) = \nabla\phi.d\bar{r}$

DIRECTIONAL DERIVATIVE

Let ϕ(x,y,z) be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is OP = r. Then the directional derivative of ϕ at P and is denoted by $d\phi/dr$.

Result 1: *The directional derivative of a scalar point function ϕ at a point P(x,y,z) in the direction of a unit vector* $\bar{e}$ *is equal to* $\bar{e}$. grad $\phi = \bar{e}.\ \nabla\phi$.

Level Surface

If a surface ϕ(x,y,z)= c be drawn through any point P($\bar{r}$), such that at each point on it, function has the same value as at P, then such a surface is called a level surface of the function ϕ through P.

e.g : equipotential or isothermal surface.

Result 2: *$\nabla\phi$ at any point is a vector normal to the level surface ϕ(x,y,z)=c through the point, where c is a constant.*

Result 3: The physical interpretation of $\nabla\phi$

The gradient of a scalar function ϕ(x,y,z) at a point *P*(x,y,z) is a **vector along the normal** to the level surface ϕ(x,y,z) = c at *P* and is in increasing direction. Its magnitude is equal to the greatest rate of increase of ϕ . *The Greatest value of directional derivative of* ϕ ***at a point P = |grad*** ϕ**|** *at the point.*

Example 1: Find the directional derivative of *f* = *xy*+*yz*+*zx* in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).

Sol:- Given *f* = *xy*+*yz*+*zx*.

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{z}\frac{\partial f}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$, then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

Directional derivative of f along the given direction $= \bar{e}.\nabla f$

$= \frac{1}{3}(i + 2j + 2k).[(y+z)i + (z+x)j + (x+y)k]\ at\ (1,2,0)$

$= \frac{1}{3}[(y+z) + 2(z+x) + 2(x+y)]\ at\ (1,2,0) = \frac{10}{3}$

Example 2: Show that $\nabla[f(r)] = \frac{f'(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol:-since $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, we have $r^2 = x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}. Similarly\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla[f(r)] = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(r) = \sum \bar{i} f'(r)\frac{\partial r}{\partial x} = \sum \bar{i} f'(r)\frac{x}{r}$$

$$= \frac{f'(r)}{r}\sum \bar{i}x = \frac{f'(r)}{r}\bar{r}$$

Note : From the above result, $\nabla(\log r) = \frac{1}{r^2}\bar{r}$

Example 3: Find a unit normal vector to the given surface $x^2y+2xz = 4$ at the point (2,-2,3).

Sol:- Let the given surface be $f = x^2y+2xz - 4$

On differentiating,

$$\frac{\partial f}{\partial x} = 2xy + 2z,\ \frac{\partial f}{\partial y} = x^2,\ \frac{\partial f}{\partial z} = 2x.$$

$$\text{grad } f = \sum \bar{i}\frac{\partial f}{\partial x} = \bar{i}(2xy + 2z) + \bar{j}x^2 + 2x\bar{k})$$

(grad f) at (2,-2,3) = $\bar{i}(-8+6) + 4\bar{j} + 4\bar{k}) = -2\bar{i} + 4\bar{j} + 4\bar{k}$

grad (f) is the normal vector to the given surface at the given point.

Hence the required unit normal vector is $\frac{\nabla f}{|\nabla f|} = \frac{2(-\bar{i} + 2\bar{j} + 2\bar{k}).}{2\sqrt{1 + 2^2 + 2^2}} = \frac{-\bar{i} + 2\bar{j} + 2\bar{k}}{3}$

Example 4: Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at (2,1,-1).

Sol: we have

$$\text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = -4\bar{i} - 4\bar{j} + 12\bar{k} \text{ at } (2,1,-1).$$

Greatest value of the directional derivative of $f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$

Example 5: If $\bar{a}$ is constant vector then prove that grad $(\bar{a}\,.\,\bar{r}) = \bar{a}$

Sol: Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, where a_1, a_2, a_3 are constants.

$\bar{a}\,.\,\bar{r} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}).(x\bar{i} + y\bar{j} + z\bar{k}) = a_1x + a_2y + a_3z$

$\frac{\partial}{\partial x}(\bar{a}.\bar{r}) = a_1, \frac{\partial}{\partial y}(\bar{a}.\bar{r}) = a_2, \frac{\partial}{\partial z}(\bar{a}.\bar{r}) = a_3$

grad $(\bar{a}\,.\,\bar{r}) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

Example 6: If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find ϕ.

Sol:- we know that $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Given that $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$

Comparing the corresponding coefficients, we have $\frac{\partial\phi}{\partial x} = yz, \frac{\partial\phi}{\partial y} = zx, \frac{\partial\phi}{\partial z} = xy$

Integrating partially w.r.t. x,y,z, respectively, we get

ϕ= xyz + a constant independent of x.

ϕ= xyz + a constant independent of y.

ϕ= xyz + a constant independent of z.

Here a possible form of ϕ is ϕ= xyz+a constant.

DIVERGENCE OF A VECTOR

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e., div $\bar{f} = \bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

hence we can write div $\bar{f}$ as div $\bar{f} = \nabla.\,\bar{f}$

This is a scalar point function.

Result 1: If the vector $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Result 2: div $(\bar{f} \pm \bar{g}) = div\,\bar{f} \pm div\,\bar{g}$

Result 3: A vector point function $\bar{f}$ is said to be $\bar{f}$ ***solenoidal*** if div $\bar{f}$ =0.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem like fluid dynamics, electricity and magnetism etc, we can interpret div $\bar{f}$ (= $\nabla.\,\bar{f}$).

1. Suppose $\bar{f}$ (x,y,z) is the velocity of a fluid at a point (x,y,z). Imagine a small rectangular box within the fluid. The divergence of $\bar{f}$ gives the rate at which the fluid flows out per unit volume at any given time. Therefore divergence of $\bar{f}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

If div $\bar{f}$ =0, then the fluid entering and leaving the element is the same, i.e., there is no change in the density of the fluid (or fluid is incompressible.)

2. The divergence of current density **J** gives the amount of charge flowing out per unit volume per second from a small element of closed surface around that point.

If div**J** =0 then it shows that the medium is free of charges.

Example 1: If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

Sol:- $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$. Then

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2) = y^2+2x^2z-6yz$

(div $\bar{f}$) at (1, -1, 1) = 1+2+6 =9

Example 2: find div $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$. Then

$$\frac{\partial \phi}{\partial x} = 3x^2 - 3yz,\ \frac{\partial \phi}{\partial y} = 3y^2 - 3zx,\ \frac{\partial \phi}{\partial z} = 3z^2 - 3xy$$

grad $\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)] + \frac{\partial}{\partial y}[3(y^2 - zx)] + \frac{\partial}{\partial z}[3(z^2 - xy)]$

$= 3(2x)+3(2y)+3(2z) = 6(x+y+z)$

Example 3: If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is ***solenoidal***, find *P*.

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1,\ \frac{\partial f_2}{\partial y} = 1,\ \frac{\partial f_3}{\partial z} = p$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 1+1+p = 2+p$

since $\bar{f}$ is solenoidal, we have div $\bar{f} = 0$. Hence p = -2.

Example 4: Find div $\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{div } \bar{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1=3$$

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function.

Then the vector function defined by $\bar{i}X\frac{\partial \bar{f}}{\partial x} + \bar{j}X\frac{\partial \bar{f}}{\partial y} + \bar{k}X\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or ∇x $\bar{f}$.

$$\text{Curl } \bar{f} = \bar{i}X\frac{\partial \bar{f}}{\partial x} + \bar{j}X\frac{\partial \bar{f}}{\partial y} + \bar{k}X\frac{\partial \bar{f}}{\partial z} = \sum\left(\bar{i}X\frac{\partial \bar{f}}{\partial x}\right)$$

Result 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then curl $\bar{f}$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\boldsymbol{i.e., curl\bar{f}} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Result 2: A vector $\bar{f}$ is said to be ***irrotational*** vector if curl $\bar{f} = \bar{o}$.

Physical Interpretation of curl

The general meaning of curl is rotation. When curl $\bar{f} = \bar{o}$, it means that no rotation is attached with the vector $\bar{f}$ whereas if curl $\bar{f}$ is non zero, it means that rotation is attached with the vector $\bar{f}$.

If $\bar{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point P(x,y,z) on the body, then $\bar{w}$ = ½ curl $\bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. Hence "***curl*** of a vector" indicates the ***rotation*** in the vector.

Example 1: if $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$. Then

$$\text{curl } \bar{f} = \nabla\text{x } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$=\bar{i}\left(\frac{\partial}{\partial y}(-3yz^2)-\frac{\partial}{\partial z}(2x^2yz)\right)+\bar{j}\left(\frac{\partial}{\partial z}(xy^2)-\frac{\partial}{\partial x}(-3yz^2)\right)+\bar{k}\left(\frac{\partial}{\partial x}(2x^2yz)-\frac{\partial}{\partial y}(xy^2)\right)$$

$=\bar{i}\left(-3z^2-2x^2z\right)+\bar{j}(0-0)+\bar{k}(4xyz-2xy)$

= curl $\bar{f}$ = at (1,-1,1) = $-\bar{i}-2\bar{k}$.

Example 2: Find curl $\bar{f}$ where $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi= x^3+y^3+z^3-3xyz$. Then

grad $\phi=\sum \bar{i}\frac{\partial\phi}{\partial x}=3(x^2-yz)\bar{i}+3(y^2-zx)\bar{j}+3(z^2-xy)\bar{k}$

curl grad $\phi=\nabla\text{x grad }\phi=3\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2-yz & y^2-zx & z^2-xy\end{vmatrix}$

$$=3[\bar{i}(-x+x)-\bar{j}(-y+y)+\bar{k}(-z+z)]=\bar{0}$$

curl $\bar{f}$ = $\bar{0}$.

Note: We can prove in general that curl (grad ϕ)=$\bar{0}$.(i.e) grad ϕ is always irrotational.

Example 3: Prove that curl $\bar{r}=\bar{0}$

Sol:- Let $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$

curl $\bar{r}=\sum\bar{i}x\frac{\partial}{\partial x}(\bar{r})=\sum(\bar{i}x\bar{i})=\bar{0}=\bar{0}$

Hence $\bar{r}$ is Irrotational vector.

Example 4: If f(r) is differentiable, show that curl { $\bar{r}$ f(r)} = $\bar{0}$ where $\bar{r}$ = $x\bar{i}+y\bar{j}+z\bar{k}$.

Sol: r = $\bar{r}=\sqrt{x^2+y^2+z^2}$ $r^2=x^2+y^2+z^2$

$\Rightarrow 2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$, similarly $\frac{\partial r}{\partial y}=\frac{y}{r}, and\ \frac{\partial r}{\partial z}=\frac{z}{r}$

curl{$\bar{r}$ f(r)}= curl{f(r)($x\bar{i}+y\bar{j}+z\bar{k}$)}= curl $(x.f(r)\bar{i}+y.f(r)\bar{j}+z.f(r)\bar{k}$

$$=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xf(r) & yf(r) & zf(r)\end{vmatrix}=\sum\bar{i}\left[\frac{\partial}{\partial y}[zf(r)]-\frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum\bar{i}\left[zf^1(r)\frac{\partial r}{\partial y}-yf^1(r)\frac{\partial r}{\partial z}\right]=\sum\bar{i}\left[zf^1(r)\frac{y}{r}-yf^1(r)\frac{z}{r}\right]$$

= $\bar{0}$.

Example 5: Find constants a, b and c if the vector $\bar{f}$ = $(2x+3y+az)\bar{i}+(bx+2y+3z)\bar{j}+(2x+cy+3z)\bar{k}$ is Irrotational.

Sol:- Given $\bar{f}=(2x+3y+az)\bar{i}+(bx+2y+3z)\bar{j}+(2x+cy+3z)\bar{k}$

$$\text{Curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = (c-3)\bar{i}-(2-a)\bar{j}+(b-3)\bar{k}$$

If the vector is Irrotational then curl $\bar{f}$ = $\bar{0}$

$\Rightarrow$c-3 = 2-a=b-3 = 0

$\Rightarrow$c=3, a=2, b=3.

Scalar potential:-

If $\bar{f}$ is Irrotational, there will always exist a scalar function φ(x,y,z) such that $\bar{f}$ =gradϕ and the scalar function φ(x,y,z) is called ***scalar potential*** of $\bar{f}$.

It is easy to prove that, if $\bar{f}$ = gradϕ, then curl $\bar{f}$ = $\bar{0}$.

Hence ∇x $\bar{f}$ = $\bar{0}$ ⇔ there exists a scalar function ϕ such that $\bar{f}$ = ∇ϕ.

Note: This idea is useful when we study the "work done by a force" later.

Example 1: Show that the vector $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f}$ = $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$

$$\text{Then curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix} = \sum i(-x+x) = \sum \bar{0} = \bar{0}$$

$\bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f}$ =∇ϕ.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z} = (x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x}=x^2-yz \Rightarrow \phi=\int\left(x^2-yz\right)dx=\frac{x^3}{3}-xyz+f_1(y,z)\text{......(1)}$$

$$\frac{\partial\phi}{\partial y}=y^2-zx \Rightarrow \phi=\frac{y^3}{3}-xyz+f_2(z,x)\text{......(2)}$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x,y)......(3)$$

From (1), (2),(3), $\phi = \frac{x^3+y^3+z^3}{3} - xyz$

$\therefore \quad \phi = \frac{1}{3}(x^3+y^3+z^3) - xyz + cons\tan t$

Which is the required scalar potential.

Laplacian Operator ∇²

We can see that $\nabla.\nabla\phi = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right) = \sum\frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$

Here the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called ***Laplacian operator***.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). If $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

Example 1: Prove that div(grad r^m)= $m(m+1)r^{m-2}$ (or) $\nabla^2(r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ then $r^2 = x^2+y^2+z^2$.

Differentiating w.r.t. 'x' partially, wet get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now grad(r^m) = $\sum \bar{i}\frac{\partial}{\partial x}(r^m) = \sum \bar{i}mr^{m-1}\frac{\partial r}{\partial x} = \sum \bar{i}mr^{m-1}\frac{x}{r} = \sum \bar{i}mr^{m-2}x$

div (grad r^m) = $\sum \bar{i}\frac{\partial}{\partial x}[mr^{m-2}x] = m\sum\left[(m-2)r^{m-3}\frac{\partial r}{\partial x}x + r^{m-2}\right]$

$= m\sum\left[(m-2)r^{m-4}x^2 + r^{m-2}\right] = m\left[(m-2)r^{m-4}\sum x^2 + \sum r^{m-2}\right]$

$= m[(m-2)r^{m-4}(r^2)+3r^{m-2}]$

$= m[(m-2)r^{m-2}+3r^{m-2}] = m[(m-2+3)r^{m-2}] = m(m+1)r^{m-2}$.

Hence $\nabla^2(r^m) = m(m+1)r^{m-2}$

Example 2: Show that $\nabla^2[f(r)] = \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f^{11}(r) + \frac{2}{r}f^1(r)$ where r = $|\bar{r}|$.

Sol: grad [f(r)] = $\nabla f(r) = \sum i\frac{\partial}{\partial x}[f(r)] = \sum i f^1(r)\frac{\partial r}{\partial x} = \sum i f^1(r)\frac{x}{r}$

$\nabla^2[f(r)]$ = div [grad $f(r)$]

$= \nabla.\nabla f(r)$

$$= \sum \frac{\partial r}{\partial x}\left[f^1(r)\frac{x}{r}\right]$$

$$= \sum \frac{r\frac{\partial}{\partial x}[f^1(r)x] - f^1(r)x\frac{\partial}{\partial x}(r)}{r^2}$$

$$= \sum \frac{r\left(f^{11}(r)\frac{\partial r}{\partial x}x + f^1(r)\right) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \sum \frac{rf^{11}(r)\frac{x}{r}x + rf^1(r) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \frac{f^{11}(r)}{r^2}\sum x^2 + \frac{1}{r}f^1(r)\sum 1 - \frac{1}{r^3}f^1(r)\sum x^2$$

$$= \frac{f^{11}(r)}{r^2}r^2 + \frac{1}{r}f^1(r)(3) - \frac{1}{r^3}f^1(r)r^2$$

$$= f^{11}(r) + \frac{2}{r}f^1(r)$$

Example 3: If ϕ satisfies Laplacian equation, show that ∇ϕ is both solenoidal and Irrotational.

Sol: given $\nabla^2\phi = 0 \Rightarrow$div(grad ϕ)= 0 ⇒ grad ϕ is solenoidal

We know that curl (grad ϕ) = $\bar{0}$⇒grad ϕ is always Irrotational.

ITVC - UNIT-V
Assignment-Cum-Tutorial Questions

SECTION-A

I) Objective Questions

1) Gradient of f(x,y,z) is
   (a) $\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$ (b) $\nabla f = \frac{\partial f}{\partial x}\bar{\imath} + \frac{\partial f}{\partial y}\bar{\jmath} + \frac{\partial f}{\partial z}\bar{k}$
   (b) (c) $\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ (d) None of these.
2) Divergence of is $\bar{F} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ is
   (a) $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ (b) $\frac{\partial f_1}{\partial x}\bar{\imath} + \frac{\partial f_2}{\partial y}\bar{\jmath} + \frac{\partial f_3}{\partial z}\bar{k}$ (c) $\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}$ (d) $f_1 + f_2 + f_3$
3) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ then $\nabla.\bar{r} =$
   (a) 1 (b) 2 (c) 3 (d) 0
4) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ then $\nabla X\bar{r} =$
   (a) $\bar{\imath}$ (b) $\bar{\jmath}$ (c) $\bar{k}$ (d) $\bar{o}$

5) Given that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$, which of the following is false?

(a) $\nabla r^n = nr^{n-1}\bar{r}$ (b) $\nabla \sin(r) = cosr\frac{\bar{r}}{r}$ (c) $\nabla\frac{1}{r} = -\frac{\bar{r}}{r^3}$ (d) $\nabla logr = \frac{\bar{r}}{r}$

6) The Divergence of $\bar{F} = -sin\theta\bar{i} + cos\theta\bar{j}$ is

(a) 0 (b) $-cos\theta - sin\theta$ (c) $\frac{sin2\theta}{r}$ (d) $-cos\theta\bar{i} - sin\theta\bar{j}$

7) The Curl of $\bar{F} = x^3y^2\bar{i} - 3x^2y\bar{j} + xyz\bar{k}$ is

(a) $xz\bar{i} - yz\bar{j} + (-6xy + 2x^3y)\bar{k}$ (b) $3x^2y^2\bar{i} - 3x^2\bar{j} + xy\bar{k}$

(c) $xz\bar{i} - yz\bar{j} + (-6xy + 2x^3y)\bar{k}$ (d) $3x^2y^2\bar{i} - 3x^2\bar{j} + xy\bar{k}$

8) If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find ϕ.

9) Find constants a,b,c so that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is Irrotational.

10) If $\bar{a}$ is constant vector then prove that grad $(\bar{a}.\bar{r}) = \bar{a}$

SECTION-B

II) Descriptive Questions

1) Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2, 3).
2) Find the directional derivative of $2xy+z^2$ at (1,-1,3) in the direction of $\bar{i} + 2\bar{j} + 3\bar{k}$.
3) Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.
4) Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at (2,1,-1).
5) Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$.
6) If ω is a constant vector, evaluate curl V where $V = \omega x \bar{r}$.
7) If $f = (x^2+y^2+z^2)^{-n}$ then find div grad f and determine n if div grad $f = 0$.
8) Find the directional derivative of $\phi(x,y,z) = x^2yz+4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface $f(x,y,z) = x \log z - y^2$ at (-1,2,1).
9) Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).
10) If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change.
11) Evaluate the angle between the normals to the surface $xy = z^2$ at the points (4,1,2) and (3,3,-3).
12) Find the angle of intersection of the spheres $x^2+y^2+z^2 = 29$ and $x^2+y^2+z^2+4x-6y-8z-47 = 0$ at the point (4,-3,2).
13) Find the values of a and b so that the surfaces $ax^2-byz = (a+2)x$ and $4x^2y+z^3 = 4$ may intersect orthogonally at the point (1, -1,2).(or) Find the constants a and b so that surface $ax^2-byz=(a+2)x$ will orthogonal to $4x^2y+z^3=4$ at the point (1,-1,2).

SECTION-C

C. Questions testing the analyzing / evaluating ability of students

1. If $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}.\bar{r})$.
2. Verify if ∇x$\left(\frac{\bar{A}X\bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}.\bar{A})\bar{r}}{r^{n+2}}$.
3. The magnitude of the gradinat of the function f = xyz^3 at (1,0,2) is
   a) 0 b) 3 c) 8 d) α
4. For the function $\phi = ax^2y - y^3$ to represent the velocity potential of an ideal fluid $\nabla^2\phi$ should be equal to zero J_n that case, the value of a has to be
   a) -1 b) 1 c) -3 d) 3
5. If $\bar{V} = 2xy\bar{i} + (2y^2 - x^2)\bar{j}$, the velocity vector, curl $\bar{v}$ will be
   a) $2y^2\bar{j}$ b) 6y$\bar{k}$ c) Zero d) $-4x\bar{k}$ **(GATE 1997)**
6. The maximum value of the directional derivative of the function $\nabla\phi = 2x^2 + 3y^2 + 5z^2$ at (1,1,-1) is **(GATE 2000)**
   a) 10 b) -4 c) $\sqrt{152}$ d) 152
7. The divergence of the vector field $(x-y)\bar{i} + (y-x)j + (x+y+z)\bar{k}$ is **(GATE 2008)**
   a) 0 b) 2 c) 1 d) 3

INTEGRAL TRANSFORMS AND VECTOR CALCULUS

UNIT – VI

Vector Integration

Objectives:

- To Provide geometric and physical explanation of the integral of vector field over a curve.
- To apply the vector integral theorems to evaluate line, surface and volume integrals.

Syllabus:

Vector Integration:

Line, surface and volume integrals. Integral theorems: Greens - Stokes - Gauss Divergence Theorems (Without proof) and related problems. Applications: Work done, flux across the surface.

Sub Outcomes:

- Use line integrals to evaluate arc length, workdone by a vector field.
- Apply greens theorem to evaluate line integrals.
- Examine path dependence/independence of line integrals of vector field.
- Apply stokes and divergence theorems to evaluate surface and volume integrals.

Learning Material

Line Integral:

Any Integral which is evaluated along the curve is called Line Integral, and it is denoted by $\int_C \overline{F}.d\overline{r}$ where $\overline{F}$ is a vector point function, $\overline{r}$ is position vector and C is the curve.

Circulation:

If $\overline{v}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \overline{v}.d\overline{r}$ is called the circulation of $\overline{v}$ round the curve C.

Work done by a force:

- Work done by a force $\overline{F}$ during displacement from A to B is given by $\int_A^B \overline{F}.d\overline{r}$.

Q. If $\overline{F}(x,y,z) = x^3\overline{i} + y\overline{j} + z\overline{k}$ is the force field. Find the work done by $\overline{F}$ along the line from (1, 2, 3) to (3, 5, 7).

Solution The given line is $\dfrac{x-1}{3-1} = \dfrac{y-2}{5-2} = \dfrac{z-3}{7-3} = t$ (say)

$\therefore \quad x = 2t + 1,\ y = 3t + 2,\ z = 4t + 3$

Now, $\quad \overline{r} = x\hat{i} + y\hat{j} + z\hat{k} = (2t+1)\hat{i} + (3t+2)\hat{j} + (4t+3)\hat{k}$

$\therefore \quad d\overline{r} = (2\hat{i} + 3\hat{j} + 4\hat{k})\,dt$

- At $A(1, 2, 3)$, $t = 0$ and at $(3, 5, 7)$, $t = 1$

$\therefore$ Work done $= \int_C \overline{F}\cdot d\overline{r} = \int_0^1 [(2t+1)^3\,\hat{i} + (3t+2)\,\hat{j} + (4t+3)\hat{k}]$

$\cdot (2\hat{i} + 3\hat{j} + 4\hat{k})\,dt$

$$= \int_0^1 [2(2t+1)^3 + 3(3t+2) + 4(4t+3)]\,dt$$

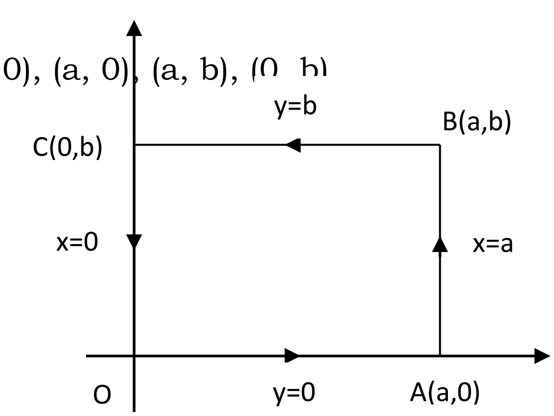
$$= \left[4t^4 + 8t^3 + \frac{37}{2}t^2 + 20t\right]_0^1$$

$$= 4 + 8 + \frac{37}{2} + 20 = \frac{101}{2}$$

- **Q.** Evaluate $\oint_C (x^2 + y^2)dx - 2xydy$ where C is a rectangle with vertices (0, 0), (a, 0), (a, b), (0, b).

Solution: Draw the given rectangle with vertices (0, 0), (a, 0), (a, b), (0, b)

Given integral $\oint_C (x^2 + y^2)dx - 2xydy$



$$\oint_C (x^2+y^2)dx-2xydy=\int_{OA}(x^2+y^2)dx-2xydy\int_{AB}(x^2+y^2)dx-2xydy+\int_{BC}(x^2+y^2)dx-2xydy$$
$$+\int_{CO}(x^2+y^2)dx-2xydy$$

Along OA: $y=0 \Rightarrow dy=0$

x varies from 0 to a

$$\int_{OA}(x^2+y^2)dx-2xydy=\int_0^a x^2dx=\frac{a^3}{3}$$

Along AB: $x=a \Rightarrow dx=0$

y varies from 0 to b

$$\int_{AB}(x^2+y^2)dx-2xydy=\int_0^b -2ady=-ab^2$$

Along BC: $y=b \Rightarrow dy=0$

x varies from a to 0

$$\int_{BC}(x^2+y^2)dx-2xydy=\int_a^0(x^2+b^2)dx=-\frac{a^3}{3}-ab^2$$

Along CO: $x=0 \Rightarrow dx=0$

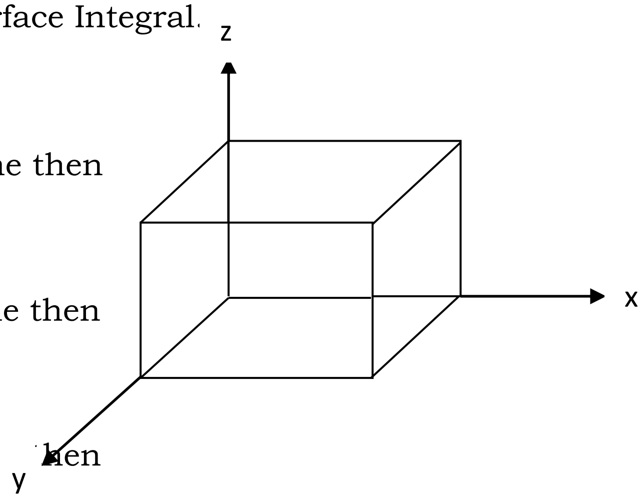
y varies from b to 0

$$\int_{CO}(x^2+y^2)dx-2xydy=\int_b^0 0dy=0$$

$$\therefore \oint_C (x^2+y^2)dx-2xydy=\frac{a^3}{3}-ab^2-\frac{a^3}{3}-ab^2+0$$
$$\therefore \oint_C (x^2+y^2)dx-2xydy=-2ab^2$$

Surface Integral :

- The Integral which is evaluated over a surface is called Surface Integral.
- If S is any surface and $\bar{n}$ is the outward drawn unit normal vector to the surface S then $\int_S \bar{F}.\bar{n}ds$ is called the Surface Integral.



Note:

- Let R_1 be the projection of S on xy-plane then
  $$\int_S \bar{F}.\bar{n}ds=\iint_{R_1}\frac{\bar{F}.\bar{n}dxdy}{|\bar{n}.\bar{k}|}$$
- Let R_2 be the projection of S on yz-plane then
  $$\int_S \bar{F}.\bar{n}ds=\iint_{R_2}\frac{\bar{F}.\bar{n}dydz}{|\bar{n}.\bar{i}|}$$
- Let R_3 be the projection of S on zx-plane then

$$\int_S \overline{F}.\overline{n}ds = \iint_{R_3} \frac{\overline{F}.\overline{n}dzdx}{|\overline{n}.\overline{j}|}$$

Q. Evaluate $\iint_S (yz\overline{i} + zx\overline{j} + xy\overline{k}).d\overline{S}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

Solution We have $\phi = x^2 + y^2 + z^2 - a^2$. Then,

$$\nabla\phi = \hat{i}\cdot 2x + \hat{j}\cdot 2y + \hat{k}\cdot 2z \text{ and } |\nabla\phi| = 2a$$

$\therefore$ The unit normal vector to the surface ϕ is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

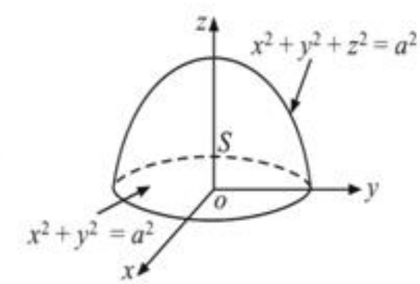
$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2a} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

Also, $$\bar{F}\cdot\hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right) = \frac{xyz + xyz + xyz}{a}$$

$$= \frac{3xyz}{a} \qquad \text{(i)}$$

$\therefore$ $$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R \bar{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \qquad \text{(ii)}$$

where R is the projection of S on xy-plane. This projection R is bounded by the x-axis, y-axis and the circle $x^2 + y^2 = a^2$, $z = 0$. Hence, x varies from 0 to a and y from 0 to $\sqrt{a^2 - x^2}$ (Figure 5.8).



Therefore, $$|\hat{n}\cdot\hat{k}| = \left|\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\cdot\hat{k}\right|$$

$$= \frac{z}{a}$$

Hence, Eq. (ii) becomes

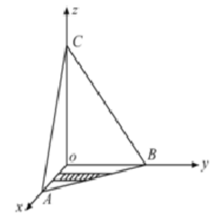
$$\iint_S \bar{F}\cdot\hat{n}dS = \iint_R \frac{3xyz}{a}\,\frac{dx\,dy}{(z/a)} = 3\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= 3\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \frac{3}{2}\int_0^a x(a^2 - x^2)\,dx$$

$$= \frac{3}{2}\left[\frac{x^2a^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{3a^4}{8}$$

Q. Evaluate $\iint_S \bar{F}.\bar{n}ds$, where $\bar{F} = 6z\bar{i} - 4\bar{j} + y\bar{k}$ and S is the portion of the plane $2x + 3y + 6z = 12$, which is in the first octant.

Solution The surface is $\phi = 2x + 3y + 6z - 12$.



$$\therefore \quad \nabla\phi = \hat{i}\cdot 2 + \hat{j}\cdot 3 + \hat{k}\cdot 6$$

and $$|\nabla\phi| = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$\therefore \quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}$$

$$\bar{F}\cdot\hat{n} = (6z\hat{i} - 4\hat{j} + y\hat{k})\cdot\frac{(2\hat{i} + 3\hat{j} + 6\hat{k})}{7} = \frac{12z - 12 + 6y}{7}$$

$$= \frac{1}{7}\left[12\,\frac{(12 - 2x - 3y)}{6} - 12 + 6y\right] = \frac{1}{7}[12 - 4x - 6y]$$

$$= \frac{1}{7}(12 - 4x)$$

Finally, $$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R \bar{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

where R is the projection of S on the xy-plane. Hence, R is bounded by x-axis, y-axis and $z = 0$. Now,

$$|\hat{n}\cdot\hat{k}| = \left|\frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}\cdot\hat{k}\right| = \frac{6}{7}$$

Therefore, $$\iint_S \bar{F}\cdot\hat{n}dS = \iint_R \frac{1}{7}(12 - 4x)\frac{dx\,dy}{(6/7)} = \iint_R \frac{2}{3}(3 - x)\,dx\,dy$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} \frac{2}{3}(3 - x)\,dy\,dx$$

$$= \int_0^6 \frac{2}{3}(3 - x)\,[y]_0^{\frac{12-2x}{3}}\,dx$$

$$= \frac{2}{3}\int_0^6 (3 - x)\left(\frac{12 - 2x}{3}\right)dx$$

$$= \frac{4}{9}\int_0^6 (18 - 6x - 3x + x^2)\,dx$$

$$= \frac{4}{9}\left[18x - \frac{9x^2}{2} + \frac{x^3}{3}\right]_0^6$$

$$= \frac{4}{9}\left[18(6) - \frac{9\times 6\times 6}{2} + \frac{(6)^3}{3}\right] = \frac{4}{9}[108 - 162 + 72]$$

$$= \frac{4}{9}[18] = 8$$

Volume Integral :

The Integral which is evaluated over a volume is called Volume Integral.

If $\overline{F}$ is a vector point function bounded by the region R with volume V, then $\int_V \overline{F}dv$ is called as Volume Integral.

Q. If $\overline{F} = 2z\overline{i} - x\overline{j} + y\overline{k}$ **,** evaluate $\iiint_V \overline{F}dV$ where V is the volume bounded by the surfaces $x = 0, \quad y = 0, \quad x = 2, \quad y = 4, \quad z = x^2, \; z = 2$

Solution. $\iiint_V \overline{F}\,dv = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\,dz \quad = \int_0^2 dx \int_0^4 dy\,[z^2\hat{i} - xz\hat{j} + yz\hat{k}]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy\,[4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2 y\hat{k}]$$

$$= \int_0^2 dx \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4 y\hat{i} + x^3 y\hat{j} - \frac{x^2y^2}{2}\hat{k}\right]_0^4$$

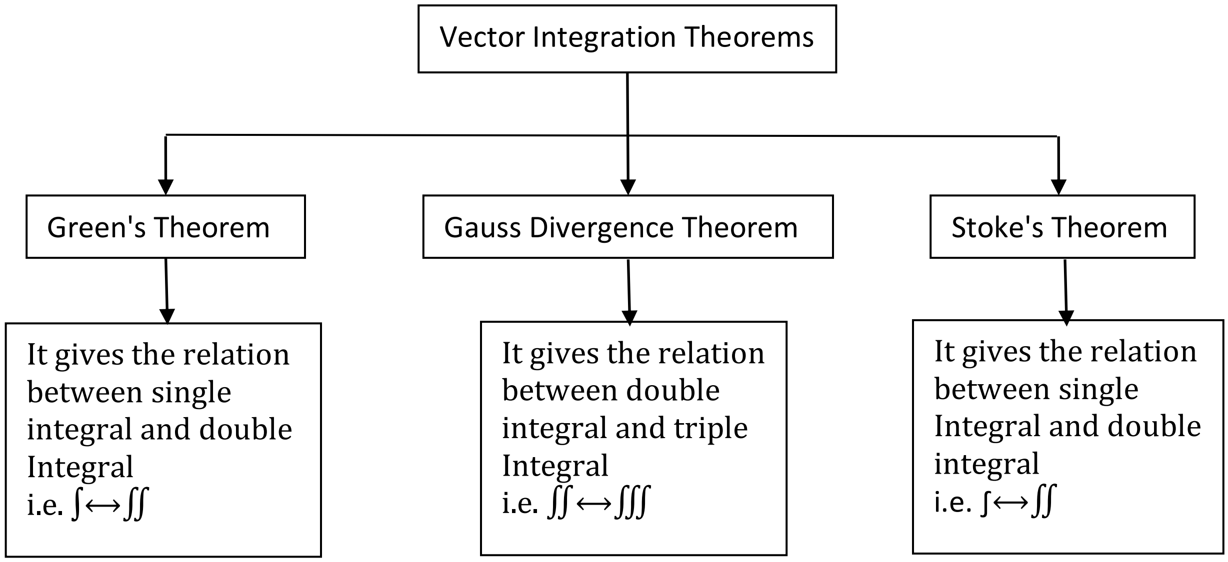
$$= \int_0^2 \; (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k})\,dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k}$$

$$= \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$

Vector Integration Theorems



Why these theorems are used?

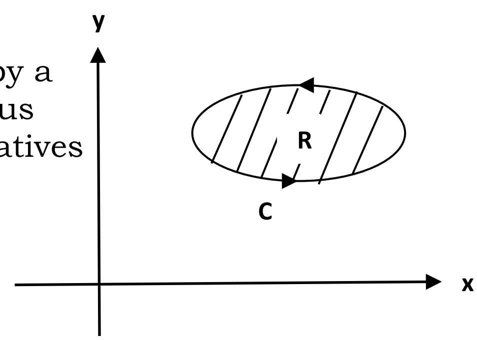
While evaluating Integration (single/double/triple) problems, we come across some Integration problems where evaluating single integration is too hard, but if we change the same problem in to double integration, the Integration problem becomes simple.

In such cases,

- We use Greens Theorem (if the given surface is xy-plane) (or) Stokes Theorem (for any plane).
- If we want to change double integration problem in to triple integral, we use Gauss Divergence Theorem.

Greens Theorem:

If R is a closed region in xy-plane bounded by a simple closed curve C and If M and N are continuous functions of x and y , and having continuous derivatives in R, then



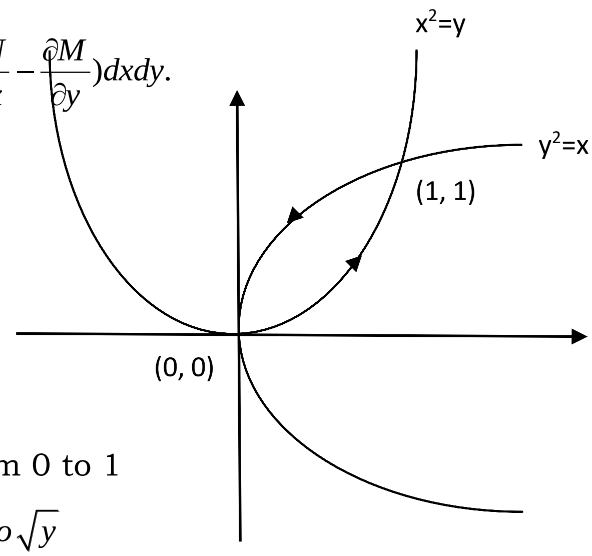
$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy.$$

Where C is traversed in the positive direction (i.e. anti clock-wise).

Note: Greens Theorem is used if the given surface is in xy -plane only.

Q. Using Green's theorem evaluate $\int_C (2xy - x^2)dx + (x^2 + y^2)dy$ where C is the closed curve of the region bounded by y= x^2 and y^2 = x.

***Solution*: Greens Theorem:** $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$



Step-1: Draw the region bounded by y= x^2 and y^2 = x.

Intersecting points are (0, 0) , (1, 1)

Step-2: Identify the limits x varies from 0 to 1

y varies from $x^2 \; to \sqrt{x}$

(Or) We can take the limits y varies from 0 to 1

x varies from $y^2 \; to \sqrt{y}$

Step-3: By comparing given integral with LHS of Greens theorem, Identify

$$M = 2xy - x^2 \qquad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

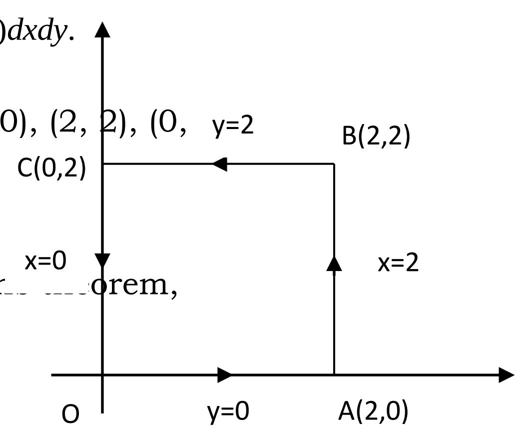
By Greens theorem $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$

$$\int_C (2xy - x^2)dx + (x^2 + y^2)dy = \int_0^1 \int_{x^2}^{\sqrt{x}} 0\,dxdy$$

$$= 0$$

Q. Verify Greens theorem in the plane for $\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy$ where C is a square with vertices (0, 0), (2, 0), (2, 2), (0, 2).

***Solution*: Greens Theorem:** $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$



Step-1: Draw the given square with vertices (0, 0), (2, 0), (2, 2), (0,

Given integral $\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy$

Step-2: By comparing given integral with LHS of Greens theorem, Identify

$$M = x^2 - xy^3 \qquad N = y^2 - 2xy$$

$$\frac{\partial M}{\partial y} = -3xy^2 \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3xy^2 - 2y$$

Evaluation of LHS: $\oint_C Mdx + Ndy$ **:**

$$\oint_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy + \int_{CO} Mdx + Ndy$$

Along OA: $y = 0 \Rightarrow dy = 0$

x varies from 0 to 2

$$\int_{OA} Mdx + Ndy = \int_{OA} (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$= \int_0^2 x^2 dx$$

$$= \frac{8}{3}$$

Along AB: $x = 2 \Rightarrow dx = 0$

y varies from 0 to 2

$$\int_{AB} Mdx + Ndy = \int_{AB} (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$= \int_0^2 (y^2 - 4y)dy$$

$$= -\frac{16}{3}$$

Along BC: $y = 2 \Rightarrow dy = 0$

x varies from 2 to 0

$$\int_{BC} Mdx + Ndy = \int_{BC} (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$= \int_2^0 (x^2 - 8x)dx$$

$$= \frac{40}{3}$$

Along CO: $x = 0 \Rightarrow dx = 0$

y varies from 2 to 0

$$\int_{CO} Mdx + Ndy = \int_{CO} (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$= \int_2^0 y^2 dy$$

$$= -\frac{8}{3}$$

$$\therefore \oint_C Mdx + Ndy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3}$$

$$\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy = 8$$

Evaluation of RHS: $\iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy$ **:**

From Region x varies from 0 to 2

y varies from 0 to 2

$$\iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy = \int_0^2 \int_0^2 (3xy^2 - 2y)dxdy$$

$$= \int_0^2 (3x\frac{y^3}{3} - 2\frac{y^2}{2})_0^2 dx$$

$$= \int_0^2 (8x - 4)dx$$

$$= 8$$

$$\therefore \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy = 8$$

$$\therefore \oint_C Mdx + Ndy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy.$$

Hence Greens theorem verified.

Gauss Divergence Theorem:

Let S be a closed surface enclosing a volume V. If $\overline{F}$ is a continuously differentiable vector point function, then

$$\int_V div\overline{F}dv = \int_S \overline{F}.\overline{n}ds$$

Where $\bar{n}$ is the outward drawn normal vector at any point of S.

Note: Let $\overline{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

Then $div\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$dv = dxdydz$

$$\iiint_V div\overline{F}dv = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

Q. Use Gauss' theorem to evaluate the surface integral $\iint_S \overline{F}.\bar{n}ds$ where F is the vector field $x^2yi + 2xyj + z^3k$ and S is the surface of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

***Solution*: Gauss Divergence Theorem:** $\int_V div\overline{F}dv = \int_S \overline{F}.\bar{n}ds$

Given vector field is

$$\overline{F} = x^2yi + 2xyj + z^3k$$

$Thus \quad F_1 = x^2y \qquad F_2 = 2xy \qquad F_3 = z^3$

$$div\overline{F} = \nabla.\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2xy + 2x + 3z^2$$

Here the limits are x varies from 0 to 1

y varies from 0 to 1

z varies from 0 to 1

By Gauss Divergence Theorem $\int_S \overline{F}.\bar{n}ds = \int_V div\overline{F}dv$

$$= \int_0^1\int_0^1\int_0^1 (2xy + 2x + 3z^2)dxdydz$$

$$= \int_0^1\int_0^1 (y + 1 + 3z^2)dydz$$

$$= \int_0^1 \left(\frac{3}{2} + 3z^2\right) dz$$

$$= \frac{11}{6}$$

Stokes Theorem:

Let S be an open surface bounded by a closed curve C. If $\overline{F}$ is any continuously differentiable vector point function then

$$\int_C \overline{F}.d\overline{r} = \iint_S Curl\,\overline{F}.\overline{n}ds$$

Where C is traversed in the positive direction and $\overline{n}$ is the outward drawn unit normal vector at any point of the surface S.

Note: Stokes Theorem is used for any surface (or) any plane (xy-plane, yz-plane, zx -plane)

Q. Using Stoke's theorem or otherwise, evaluate $\int_C [(2x-y)dx - yz^2dy - y^2zdz]$, where C is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.

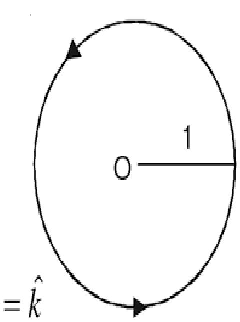
Solution. $\int_c [(2x-y)\,dx - yz^2 dy - y^2 z\,dz]$

$$= \int_c [(2x-y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}]\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

By Stoke's theorem $\oint \overline{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\overline{F}\cdot\overline{n}\,ds$ ...(1)

$$\text{Curl}\,\overline{F} = \nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2\,yz + 2\,yz)\,\hat{i} - (0-0)\,\hat{j} + (0+1)\,\hat{k} = \hat{k}$$



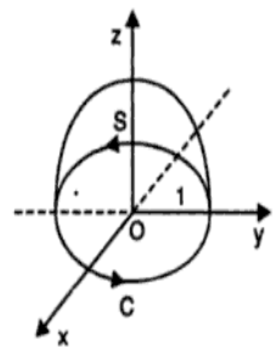
Putting the value of curl $\overline{F}$ in (1), we get

$$= \iint \hat{k}\cdot\hat{n}\,ds = \iint \hat{k}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \iint dx\,dy = \text{Area of the circle} = \pi \qquad \left[\because ds = \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}\right]$$

Q. Verify Stokes theorem for the vector field $\overline{F} = (2x - y)\overline{i} - yz^2\overline{j} - y^2z\overline{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane.

***Solution*: Stokes theorem:** $\int_C \overline{F}.d\overline{r} = \iint_S Curl\,\overline{F}.\overline{n}ds$

Evaluation of $\int_C \overline{F}.d\overline{r}$**:**



On C, $\quad x^2 + y^2 = 1$

i.e., $\quad x = \cos t,\ y = \sin t \quad \text{and} \quad z = 0$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$\vec{F}$ on $C = (2x - y)\vec{i}$

$\therefore \quad \vec{F}\cdot d\vec{r} = (2x - y)\,dx$

$$\text{L.H.S.} = \int_C \vec{F}\cdot d\vec{r} = \int_C (2x - y)\,dx$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \int_0^{2\pi} (\sin^2 t - \sin 2t)\,dt$$

$$= 4\int_0^{\pi/2} \sin^2 t\,dt - \int_0^{2\pi} \sin 2t\,dt = 4\cdot\frac{1}{2}\cdot\frac{\pi}{2} - 0 = \pi.$$

Evaluation of $\iint_S Curl\,\overline{F}.\overline{n}ds$**:**

Now, $$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \vec{k}$$

$$\text{curl } \vec{F}\,.\,\vec{n} = \vec{k}\,.\,\vec{n} = \vec{n}\,.\,\vec{k}$$

$\therefore$ $$\text{R.H.S.} = \int_S \text{curl } \vec{F}\,.\,\vec{n}\,dS = \int_S \vec{n}\,.\,\vec{k}\,dS$$

$$= \iint_{S_1} \vec{n}\,.\,\vec{k}\,\frac{dx\,dy}{\vec{n}\,.\,\vec{k}},$$ where S_1 is the projection of S on xy-plane.

$$= \iint_{S_1} dx\,dy = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$ (taking in Polar coordinates)

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{1}{2}\cdot\int_0^{2\pi} d\theta = \frac{1}{2}\,.\,2\pi = \pi$$

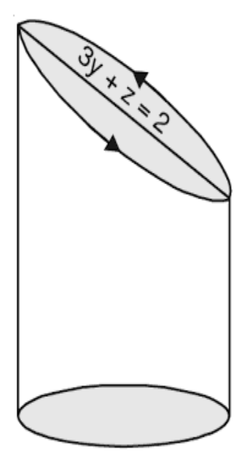
$\therefore$ L.H.S. = R.H.S.

Hence Stoke's theorem is verified.

Q. Evaluate $\int_C \overline{F}.d\overline{r}$,where $\overline{F}(x,y,z) = -y^2\overline{i} + x\overline{j} + z^2\overline{k}$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$.

Solution. $$\oint_C \vec{F}\,.\,d\vec{r} = \iint_S \text{curl } \overline{F}\,.\,\hat{n}\,ds = \iint_S \text{curl}\,(-y^2\,\hat{i} + x\,\hat{j} + z^2\,\hat{k})\,\hat{n}\,ds \qquad ...(1)$$

$$F(x, y, z) = -y^2\,\hat{i} + x\,\hat{j} + z^2\,\hat{k}$$ *(By Stoke's Theorem)*



$$\text{Curl } \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

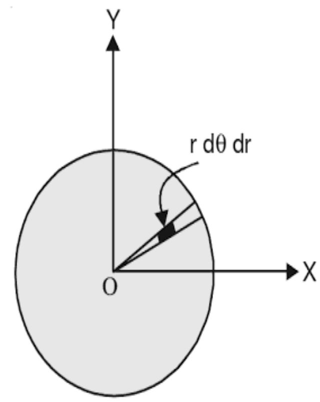
$$= \hat{i}\,(0-0) - \hat{j}\,(0-0) + \hat{k}\,(1+2y) = (1+2y)\,\hat{k}$$

Normal vector = $\nabla\,\overline{F}$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(y+z-2) = \hat{j} + \hat{k}$$

Unit normal vector $\hat{n} = \dfrac{\hat{j}+\hat{k}}{\sqrt{2}}$

$$ds = \frac{dx\,dy}{\hat{\eta}\cdot\hat{k}}$$



On putting the values of curl $\vec{F}, \hat{n}$ and ds in (1), we get

$$\int_C \vec{F}\cdot\vec{dr} = \iint_S (1+2y)\,\hat{k}\cdot\frac{\hat{j}+\hat{k}}{\sqrt{2}}\,\frac{dx\,dy}{\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right)\cdot\hat{k}}$$

$$= \iint \frac{1+2y}{\sqrt{2}}\,\frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint (1+2y)\,dx\,dy = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^1 (r+2r^2\sin\theta)\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta\left[\frac{r^2}{2}+\frac{2r^3}{3}\sin\theta\right]_0^1 = \int_0^{2\pi}\left[\frac{1}{2}+\frac{2}{3}\sin\theta\right]d\theta$$

$$= \left[\frac{\theta}{2}-\frac{2}{3}\cos\theta\right]_0^{2\pi} = \left(\pi-\frac{2}{3}-0+\frac{2}{3}\right) = \pi$$

Q. Verify Stokes theorem for the vector field $\bar{F} = (x^2-y^2)\bar{i} + 2xy\bar{j}$ integrated round the rectangle in the plane z=0 and bounded by the lines *x=0, y=0, x=a, y=b.*

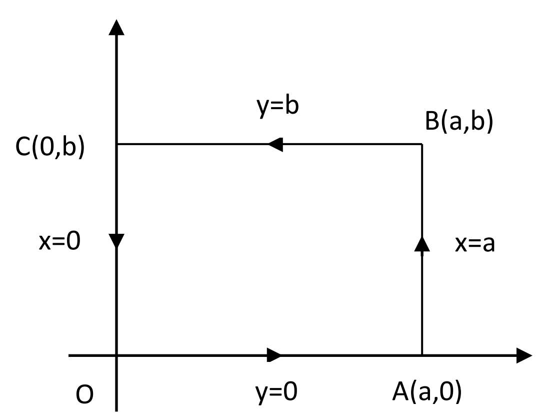
***Solution*: Stokes theorem:** $\int_C \bar{F}.d\bar{r} = \iint_S Curl\bar{F}.\bar{n}ds$

Evaluation of LHS=$\int_C \bar{F}.d\bar{r}$**:**

Draw the given rectangle in the plane z=0 and bounded by the lines *x=0, y=0, x=a, y=b.*

i.e rectangle with vertices (0, 0), (a, 0), (a, b), (0, b).

$$\oint_C \bar{F}.d\bar{r} = \int_{OA}\bar{F}.d\bar{r} + \int_{AB}\bar{F}.d\bar{r} + \int_{BC}\bar{F}.d\bar{r} + \int_{CO}\bar{F}.d\bar{r}$$



We know that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Thus $\bar{F}.d\bar{r} = (x^2-y^2)dx + 2xydy$

Along OA: $y = 0 \Rightarrow dy = 0$

x varies from 0 to a

$$\int_{OA} \bar{F}.d\bar{r} = \int_{OA} (x^2 - y^2)dx + 2xydy = \int_0^a x^2 dx = \frac{a^3}{3}$$

Along AB: $x = a \Rightarrow dx = 0$

y varies from 0 to b

$$\int_{AB} \bar{F}.d\bar{r} = \int_{AB} (x^2 - y^2)dx + 2xydy = \int_0^b 2ady = ab^2$$

Along BC: $y = b \Rightarrow dy = 0$

x varies from a to 0

$$\int_{BC} \bar{F}.d\bar{r} = \int_{BC} (x^2 - y^2)dx + 2xydy = \int_a^0 (x^2 - b^2)dx = -\frac{a^3}{3} + ab^2$$

Along CO: $x = 0 \Rightarrow dx = 0$

y varies from b to 0

$$\int_{CO} \bar{F}.d\bar{r} = \int_{CO} (x^2 - y^2)dx + 2xydy = \int_b^0 0dy = 0$$

$$\therefore \oint_C \bar{F}.d\bar{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$\therefore \oint_C \bar{F}.d\bar{r} = 2ab^2$$

Evaluation of RHS= $\iint_S Curl\bar{F}.\bar{n}ds$

Now, $$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\,\vec{k}$$

For the surface, $S, \vec{n} = \vec{k}$

$\therefore$ $$\text{curl } \vec{F} \cdot \vec{n} = 4y$$

$$\text{R.H.S.} = \iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS = \int_0^a \int_0^b 4y\, dy\, dx$$

$$= \int_0^a 4 \cdot \left[\frac{y^2}{2}\right]_0^b dx = 2b^2 \int_0^a dx = 2ab^2$$

$\therefore$ L.H.S. = R.H.S. Hence the Stoke's theorem is verified

Assignment-Cum-Tutorial Questions

SECTION-A

I) Objective Questions

1. $Evaluate \int_C \bar{F}.d\bar{r}$,where $\bar{F} = (x+y)\bar{i} + (y-x)\bar{j}$ and C is
   (i) the parabola $y^2=x$ between the points (1, 1) and (4, 2).
   (ii) the straight line joing the points (1, 1) and (4, 2).
2. The value of $\int_C \bar{F}.d\bar{r}$, where $\bar{F} = i + j + k$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $x = t, y = t, z = t, \quad 0 \le t \le 1$ is the curve C.
3. Use Green's Theorem to evaluate $\int_C y^2dx + xydy$ for C: boundary of the region lying between the graphs of $y = 0, y = \sqrt{x}$ *and* $x = 9$. [ ]
   a) $-\frac{81}{2}$ b) $\frac{81}{4}$ c) $\frac{243}{4}$ d) $-\frac{81}{4}$
4. Use Stokes's Theorem to evaluate $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = y^2\bar{i} + z^2\bar{j} + 2xy\bar{k}$ around the triangle with vertices (2, 0, 0), (0, 1, 0), and (0, 0, 4) . [ ]
   a)-2 b)-6 c)2 d)6
5. If E is the solid region bounded by the planes x=0,y=0,z=0 and $2x + 2y + z = 4$ then the triple integral $\iiint_E ydV$ is [ ]
   a) $-\frac{1}{3}$ b)4 c) $\frac{4}{3}$ d) $\frac{2}{3}$
6. $\int_s \bar{r} \cdot \bar{n} \, ds =$ [ ]
   a) V b) 2V c) 3V d) $\frac{3}{2}$V
7. $\int_s (\bar{r} \times \bar{n}) \, ds =$ [ ]
   a) 0 b) r c) 1 d) -1
8. If S is any closed surface enclosing a volume V and $\bar{F} = x\,\bar{i} + 2y\,\bar{j} + 3z\,\bar{k}$ then $\iint_s \bar{F} \cdot \bar{n} \, ds =$ [ ]
   a) V b) 3V c) 6V d) 2V
9. From green's theorem $\int Pdx + Qdy =$ [ ]

a) $\iint\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right)dx\,dy$ b) $\iint\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy$ c) $\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$ d) 0

10. $\int_c (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot\bar{n}\,ds$ = [ ]

a) $\frac{2\pi}{3}(a+b+c)$ b) $\frac{4\pi}{3}(a+b+c)$ c) $(a+b+c)$

d) $\frac{3\pi}{4}(a+b+c)$

11. The work done by a force $\bar{F} = (3x^2 + 6y)\ \bar{i} - 14yz\ \bar{j} + 20xz^2\ \bar{k}$ along the lines from (0, 0, 0) to (1, 0, 0) is [ ]

a) 1/2 b) 3/2 c) 0 d) 1

12. For any closed surface S, $\iint_s \text{curl}\ \bar{F}\ \bar{n}\ ds$ = [ ]

a) 0 b) 2F c) $\bar{n}$ d) $\int \bar{F}\cdot d\bar{r}$

SECTION-B

II) Descriptive Questions

1. If $\bar{F} = xy\,\bar{i} - z\,\bar{j} + x^2\bar{k}$ and C is the Curve $x=t^2,\ y=2t,\ z=t^3$ from $t=0$ to $t=1$. Evaluate $\int_C \bar{F}.d\bar{r}$.
2. Compute the line integral $\int (y^2dx - x^2dy)$ round the triangle whose vertices are (1,0), (0,1) and (-1,0) in the xy-plane.
3. Evaluate $\int_C \bar{f}.d\bar{r}$ where $\bar{f} = x^2 i + y^2 j$ and curve c is the arc of the parabola $y=x^2$ in the xy-plane from (0,0) to (1,1).
4. Show that $\bar{F} = (2xy+z^3)i + x^2 j + 3xz^2 k$ is a conservative force field. Find the scalar potential and the work done in moving an object in this field from (1, -2, 1) to (3, 1, 4).
5. Verify divergence theorem for **F** = 4xz ***i*** – y ^{2} **j** + yz **k** taken over the cube bounded by x = 0, x = 1; y = 0 , y = 1; z = 0 , z = 1.
6. Verify Green's theorem in the xy - plane for $\oint_C [(xy^2 - 2xy)dx + (x^2y + 3)dy]$ around the boundary C of the region enclosed by y ^{2} = 8x, x = 2 and the x-axis.
7. Verify Green's theorem for $\oint_C [(xy + y^2)dx + x^2dy]$. where C is a bounded by $y=x$ and $y=x^2$.

8. Verify Stokes theorem for F = (y-z+2)i+(yz+4)j-xzk where S is the surface of the cube x=0, y=0,z=0,x=2,y=2,z=2 above the xy-plane.
9. Evaluate by Green's theorem $\oint_C [(y-\sin x)dx+\cos x dy]$ where 'C' is the triangle enclosed by the lines y = 0, x = π/2 and πy = 2x.
10. Apply Gauss divergence theorem to evaluate $\iint_S \overline{F}.\overline{n}ds$ where $F = yi + xj + zk$ and S is the surface of the cylindrical region bounded by $x^2 + y^2 = 9$ *and* $z = 0$ *and* $z = 2$.
11. Use Gauss divergence theorem to evaluate $\iint_S (yz^2 i + zx^2 j + 2z^2 k).Nds$ where S is the surface bounded by the xy-plane and the upper half of the sphere $x^2+y^2+z^2=a^2$ above this plane.
12. Evaluate the integral $I = \iint_S x^3 dydz + x^2 ydzdx + x^2 zdxdy$ using divergence theorem, where S is the surface consisting of the cylinder $x^2 + y^2 = a^2 (0 \le z \le b)$ and the circular disks z=0 and $z = b(x^2 + y^2 \le a^2)$.
13. Apply Stoke's Theorem to evaluate $\oint_C [(x+y)dx+(2x-z)dy+(y+z)dz]$ when C is the boundary of the triangle with vertices (2,0,0), (0,3,0) and (0,0,6).
14. Evaluate by Stokes theorem $\oint_C [(x+y)dx+(2x-z)dy+(y+z)dz]$, where C is the boundary of the triangle vertices (0,0,0), (1,0,0) and (1,1,0).
15. If $f = (x^2 + y - 4)i + 3xyj + (2xz + z^2)k$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = 16$. Show by using Stokes theorem that $\int_S curl\ f.nds = 2\pi a^3$

SECTION-C

C. Questions testing the analyzing / evaluating ability of students

1. If S is the surface of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $ax + by + cz = 1$. Show that $\int_S r.nds = \frac{1}{2abc}$.
2. If ϕ is a scalar point function, using Stoke's theorem prove that $Curl(grad\phi) = 0$.
3. Consider points P and Q in the xy plane with P=(1,0) and Q = (0,1) The line integral $\int_P^Q xdx + ydy$ along the semicircle with line segment PQ as its diameter **(GATE 2010)**

   a) -1 b) 0 c) 1

   d) depends on direction C clock wise or anti clock wise) of the semi circle

4. A triangle ABC consists of vertex points A(0,0) B(1,0) and C(0,1). The value of the integral $\iint 2x dx dy$ over the triangle is

(GATE 1997)

a) 1 b) $\frac{1}{3}$ c) $\frac{1}{8}$ d) $\frac{1}{9}$

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